3D from Photographs: Camera Calibration

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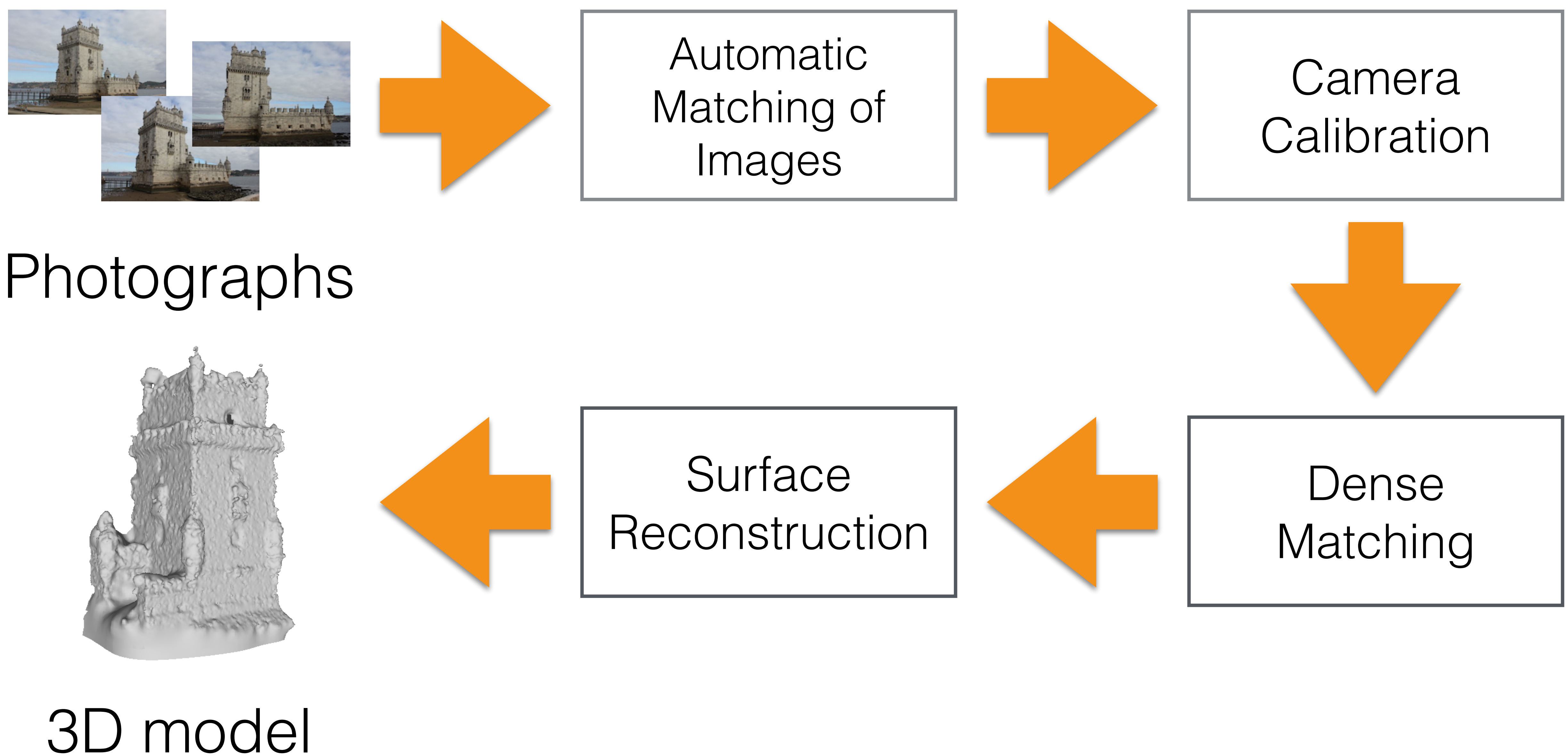
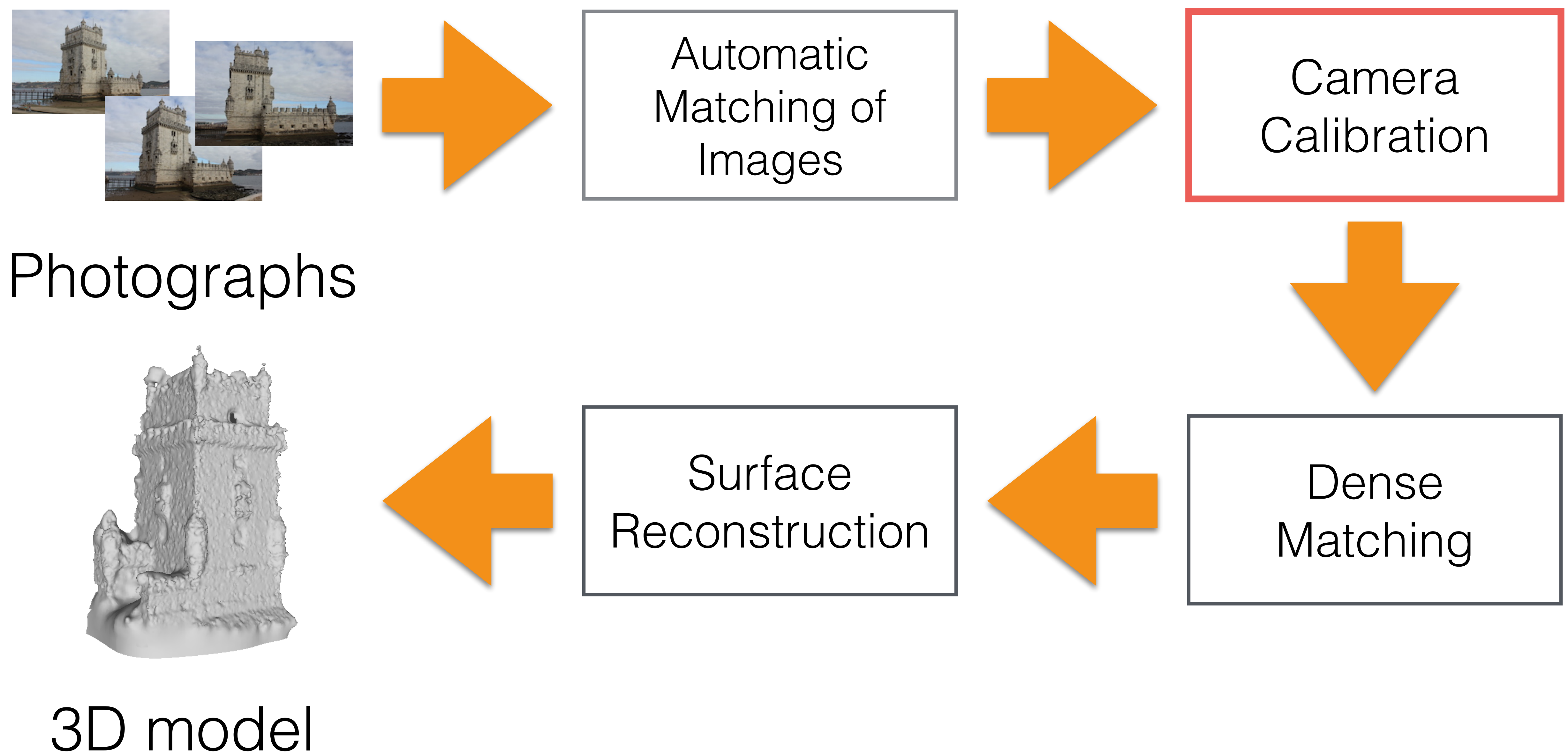
3D from Photographs

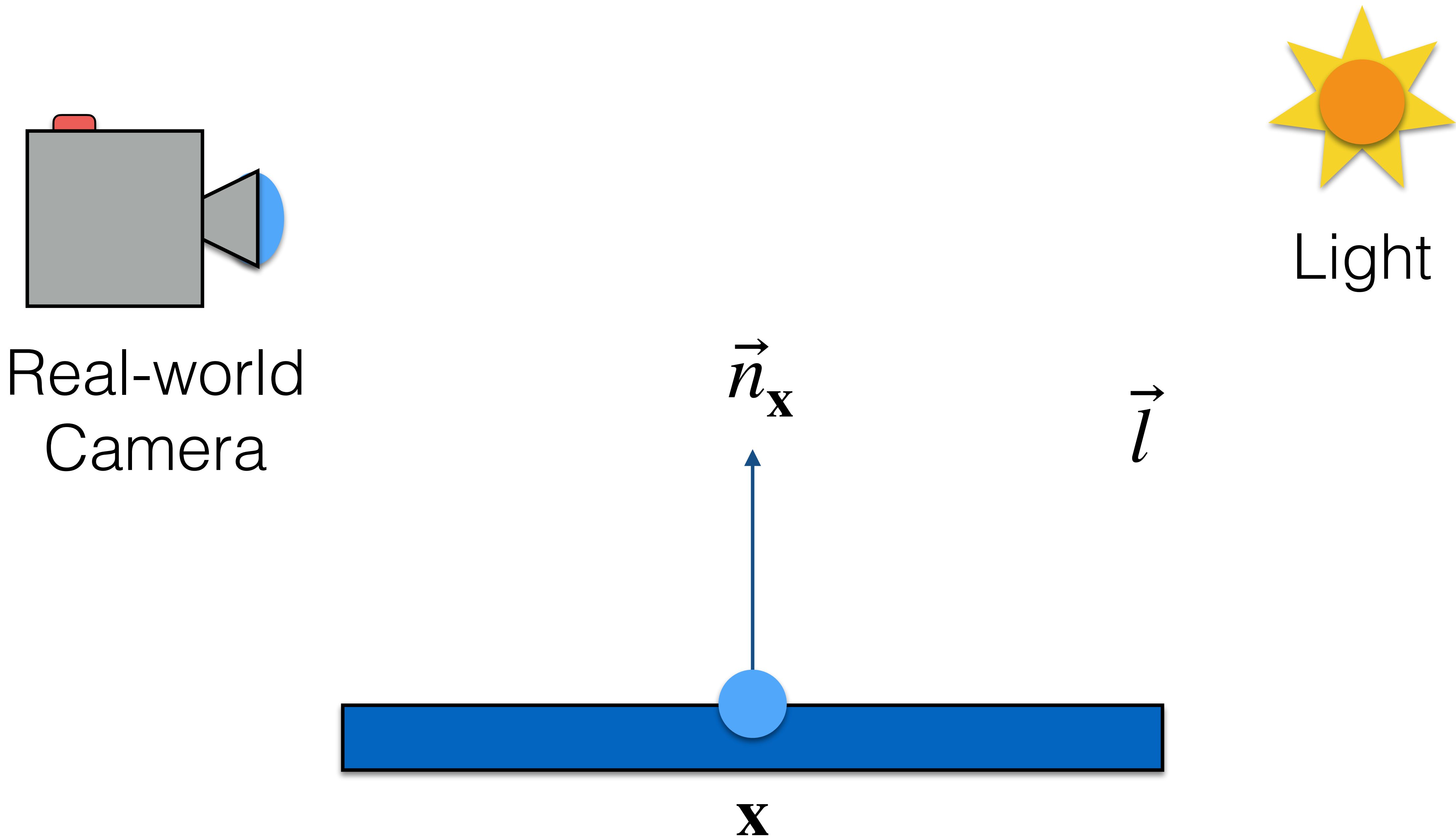


3D from Photographs

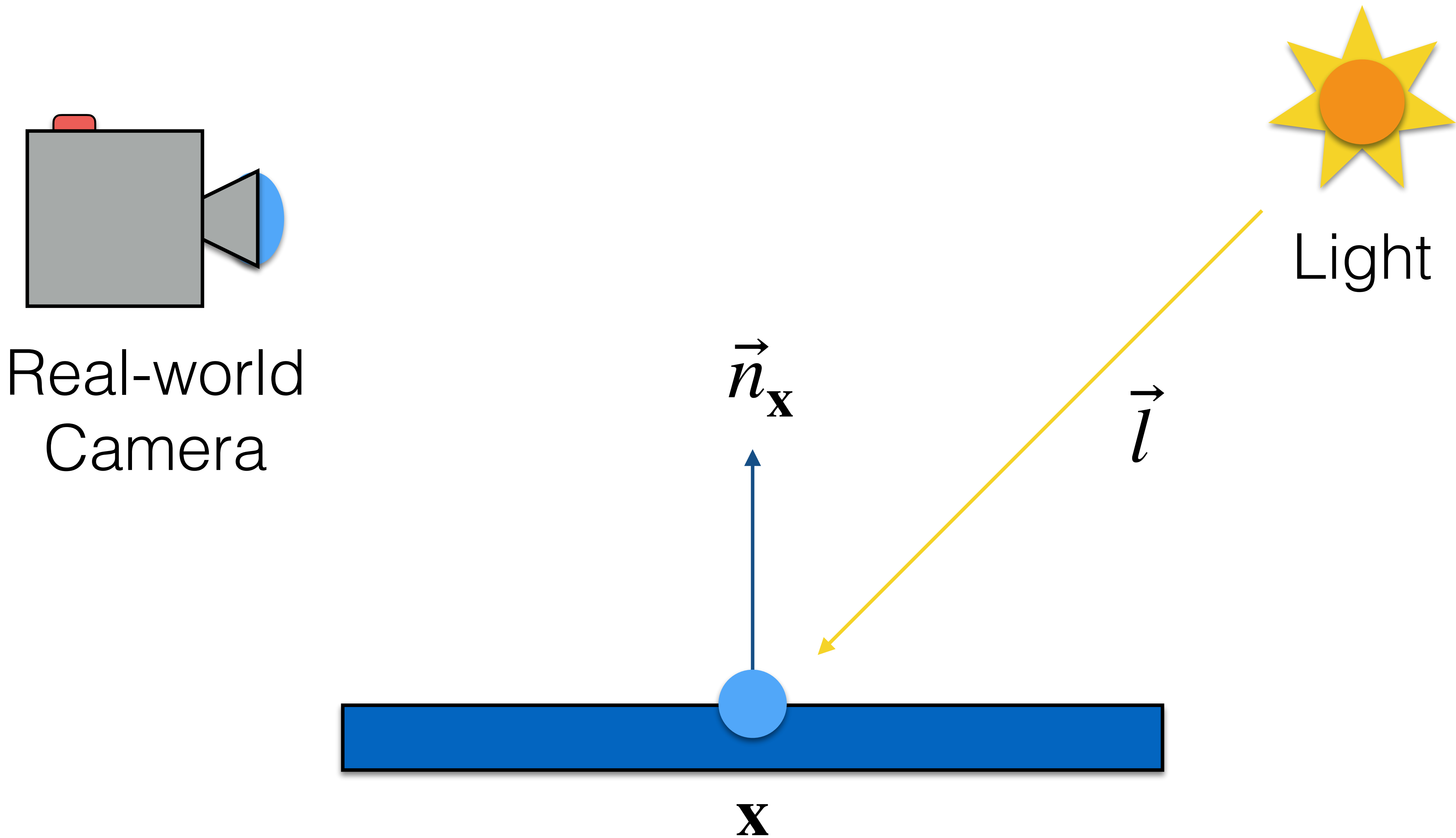


Back to the Camera Model

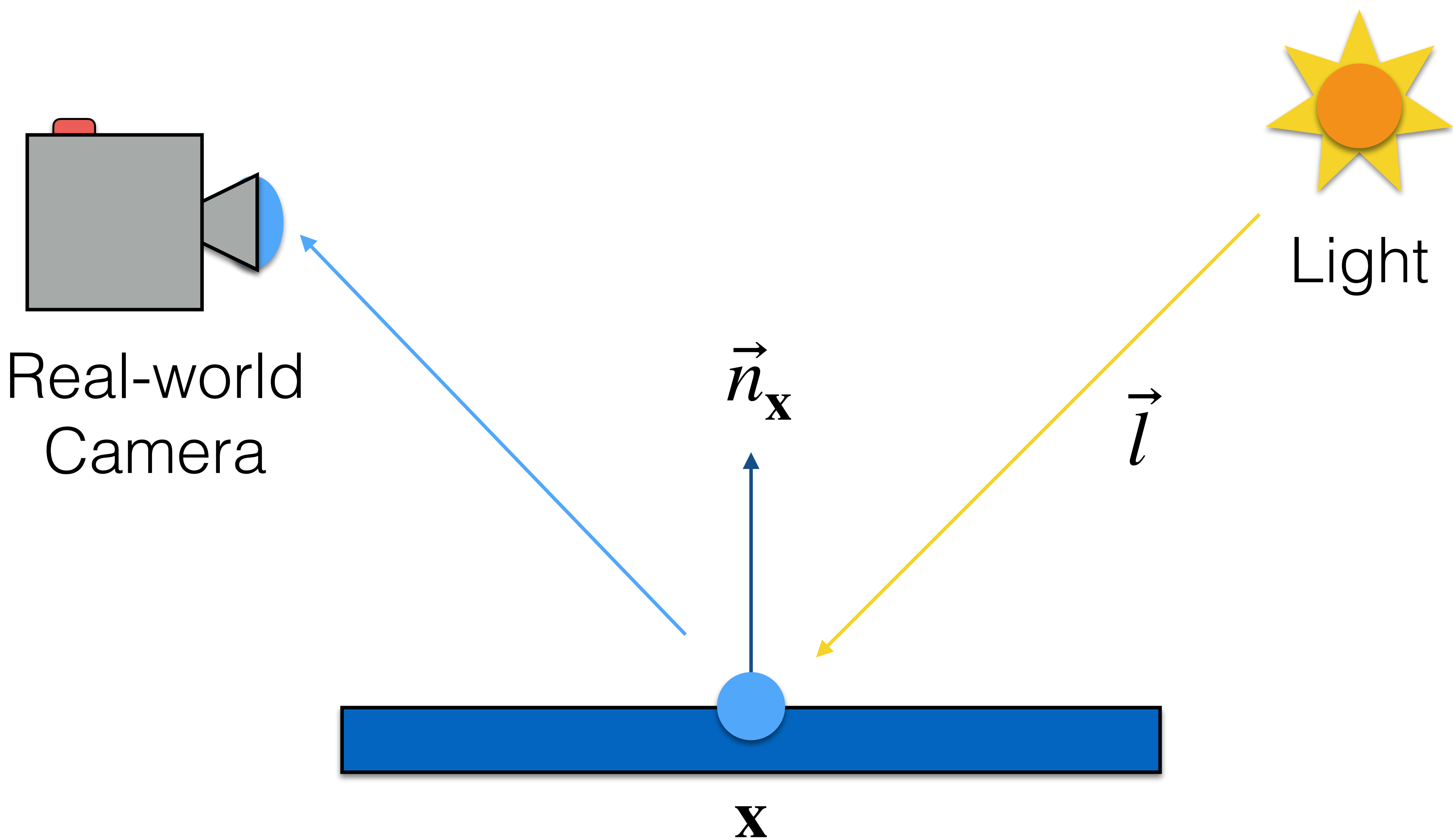
Camera Model: Image Formation



Camera Model: Image Formation



Camera Model: Image Formation



Camera Model: Pinhole Camera

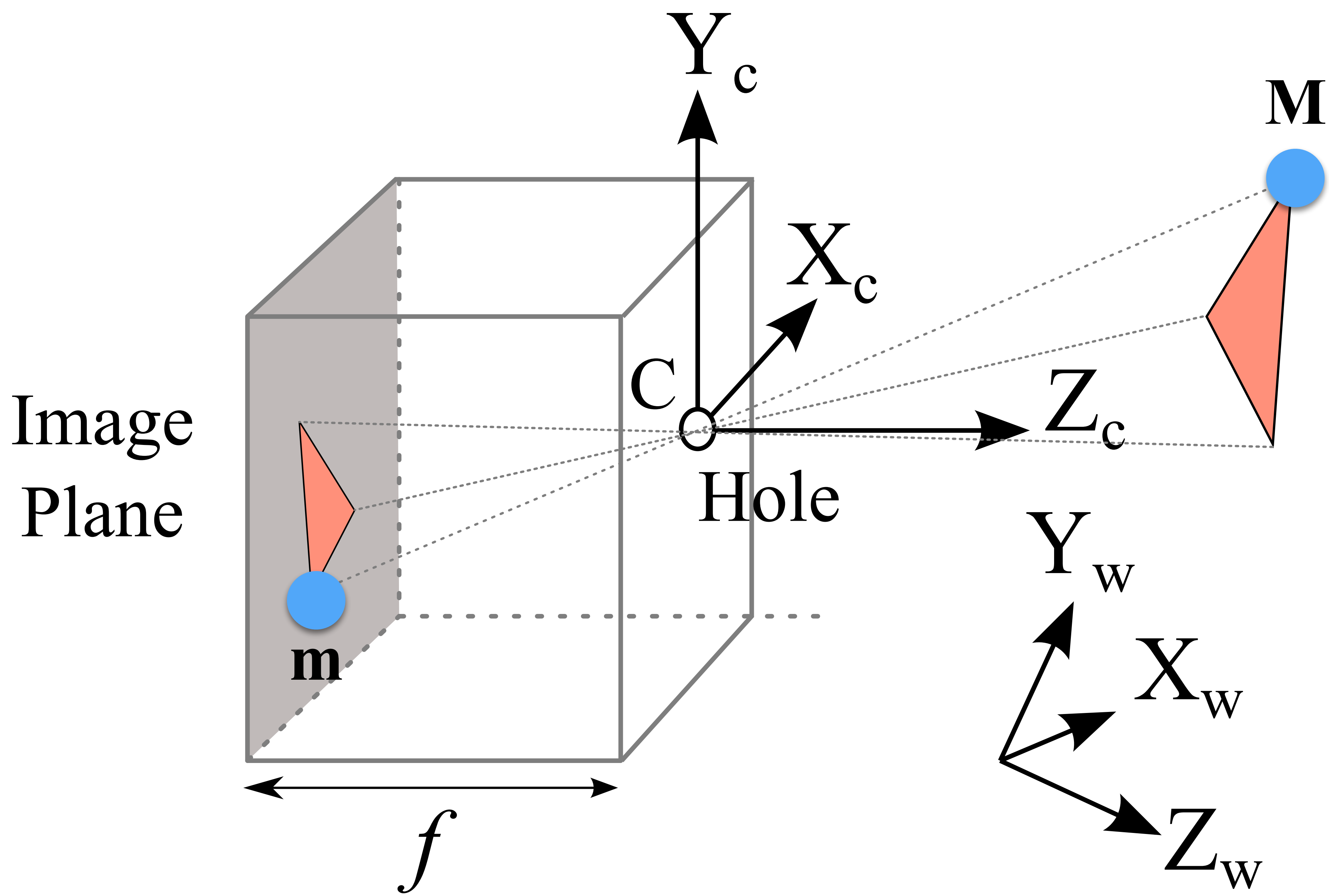
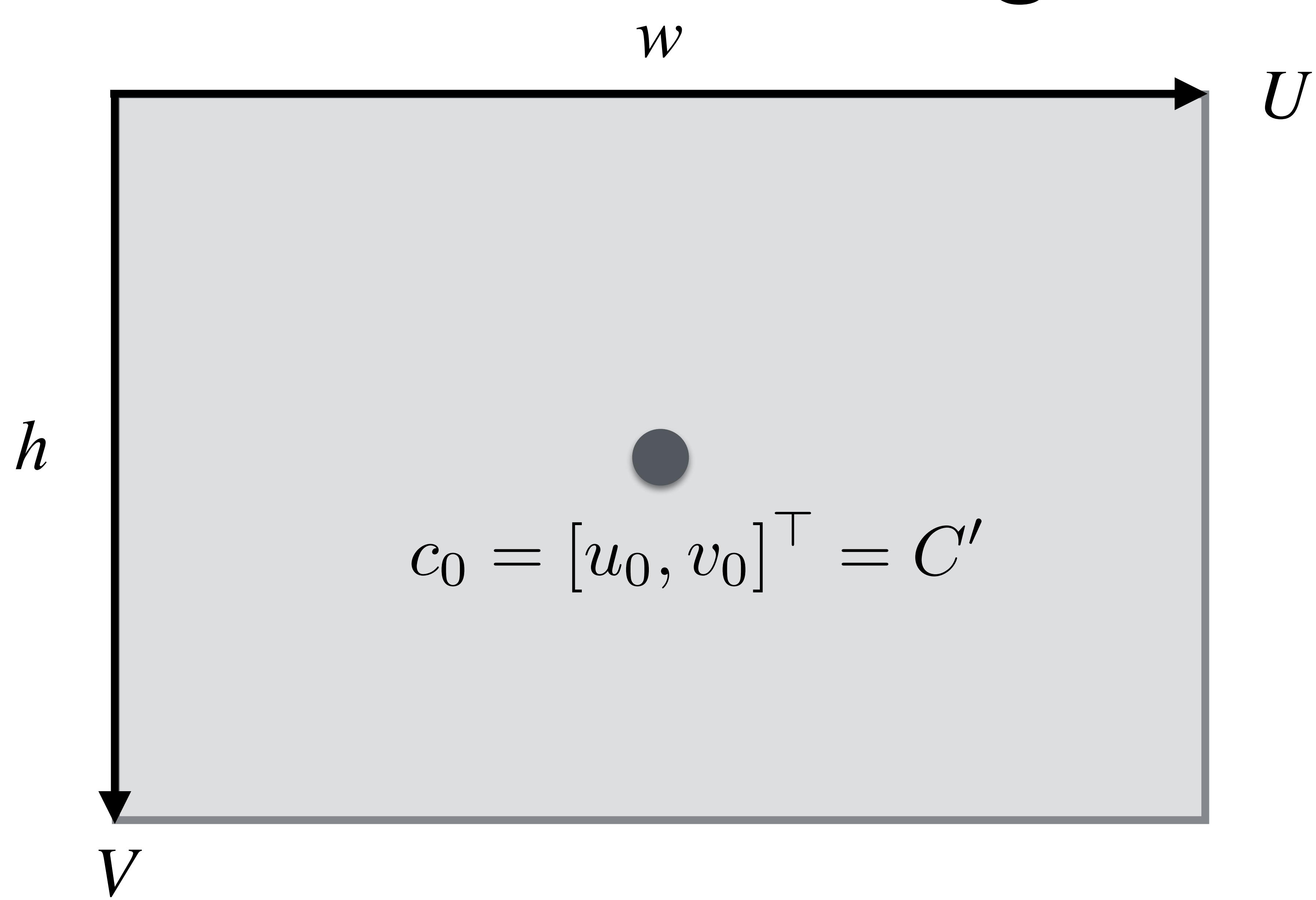


image-space

world-space

Camera Model: Image Plane



- Pixels have different height and width; i.e., (k_u, k_v).
- c_0 is called the principal point.
- The image plane has a finite size: w (width) and h (height)

Camera Model

- $\mathbf{M}$ is a point in the 3D world, and it is defined as:

$$\mathbf{M} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- $\mathbf{m}$ is a 2D point, the projection of $\mathbf{M}$, and it lives in the image plane UV:

$$\mathbf{m} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Camera Model

- By analyzing the two triangles (real-world and projected one), the following relationship emerges:

$$\frac{f}{z} = -\frac{u}{x} = -\frac{v}{y}$$

- This means that:

$$\begin{cases} u = -f \cdot \frac{x}{z} \\ v = -f \cdot \frac{y}{z} \end{cases}$$

Camera Model: Intrinsic Parameters

- If we take all into account of the optical center, and pixel size we obtain:

$$\begin{cases} u = -f \cdot \frac{x}{z} k_u + u_0 \\ v = -f \cdot \frac{y}{z} k_v + v_0 \end{cases}$$

- If we put this in matrix form, we obtain:

$$P = \begin{bmatrix} -fk_u & 0 & u_0 & 0 \\ 0 & -fk_v & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = K[I \,|\, \mathbf{0}] \quad K = \begin{bmatrix} -fk_u & 0 & u_0 \\ 0 & -fk_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{m}z = P \cdot \mathbf{M} \quad \rightarrow \quad \mathbf{m} \sim P \cdot \mathbf{M} \qquad \mathbf{m} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Camera Model: Pinhole Camera

- The perspective projection is defined as:

$$\mathbf{m} \sim P \cdot \mathbf{M}$$

$$P = K \cdot \mathbf{G} = K \cdot [R \,|\, \mathbf{t}]$$

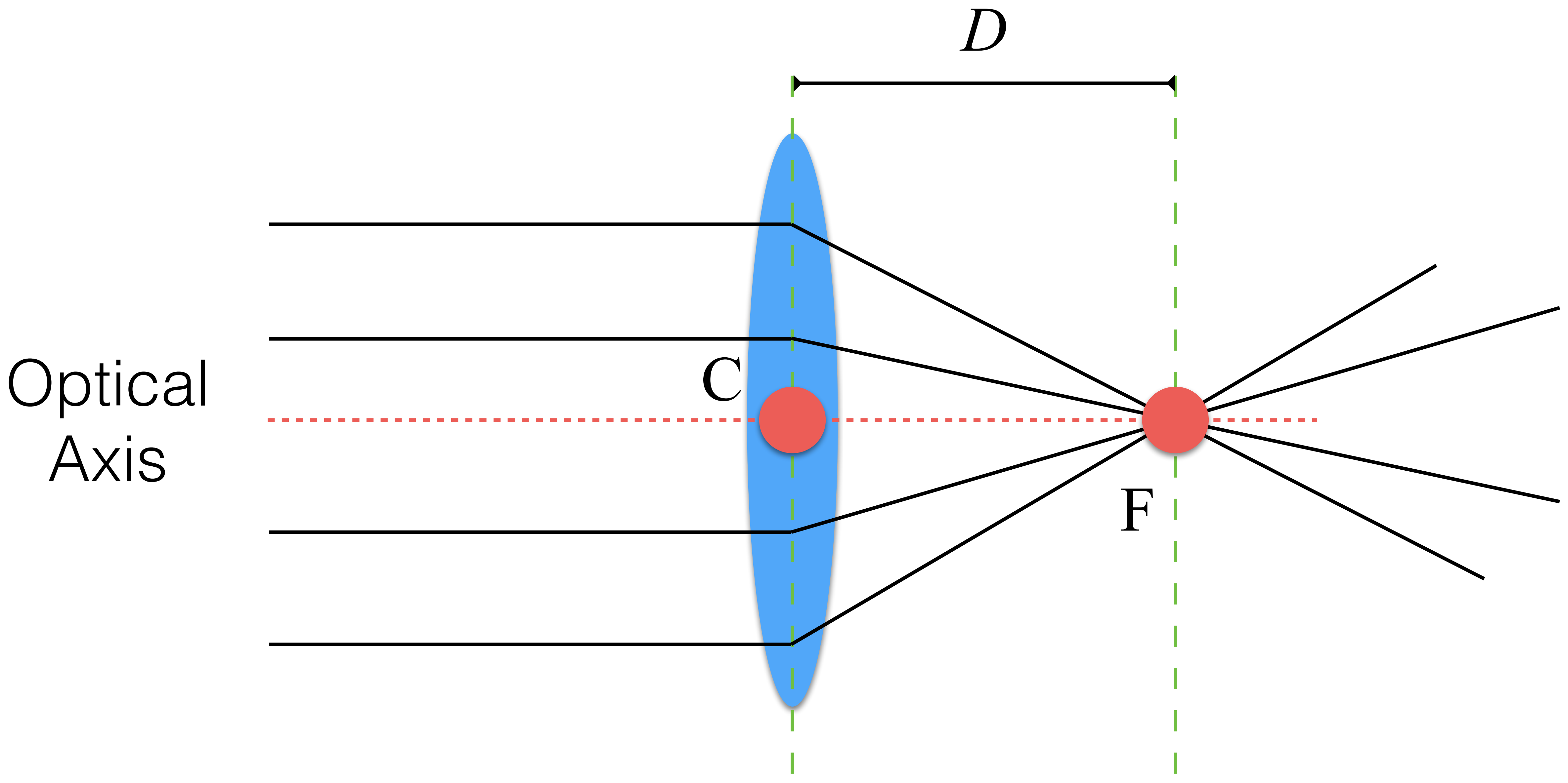
$$K = \begin{bmatrix} -fk_u & 0 & u_0 \\ 0 & -fk_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Intrinsic Matrix

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{r}_1^\top \\ \mathbf{r}_2^\top \\ \mathbf{r}_3^\top \end{bmatrix}$$

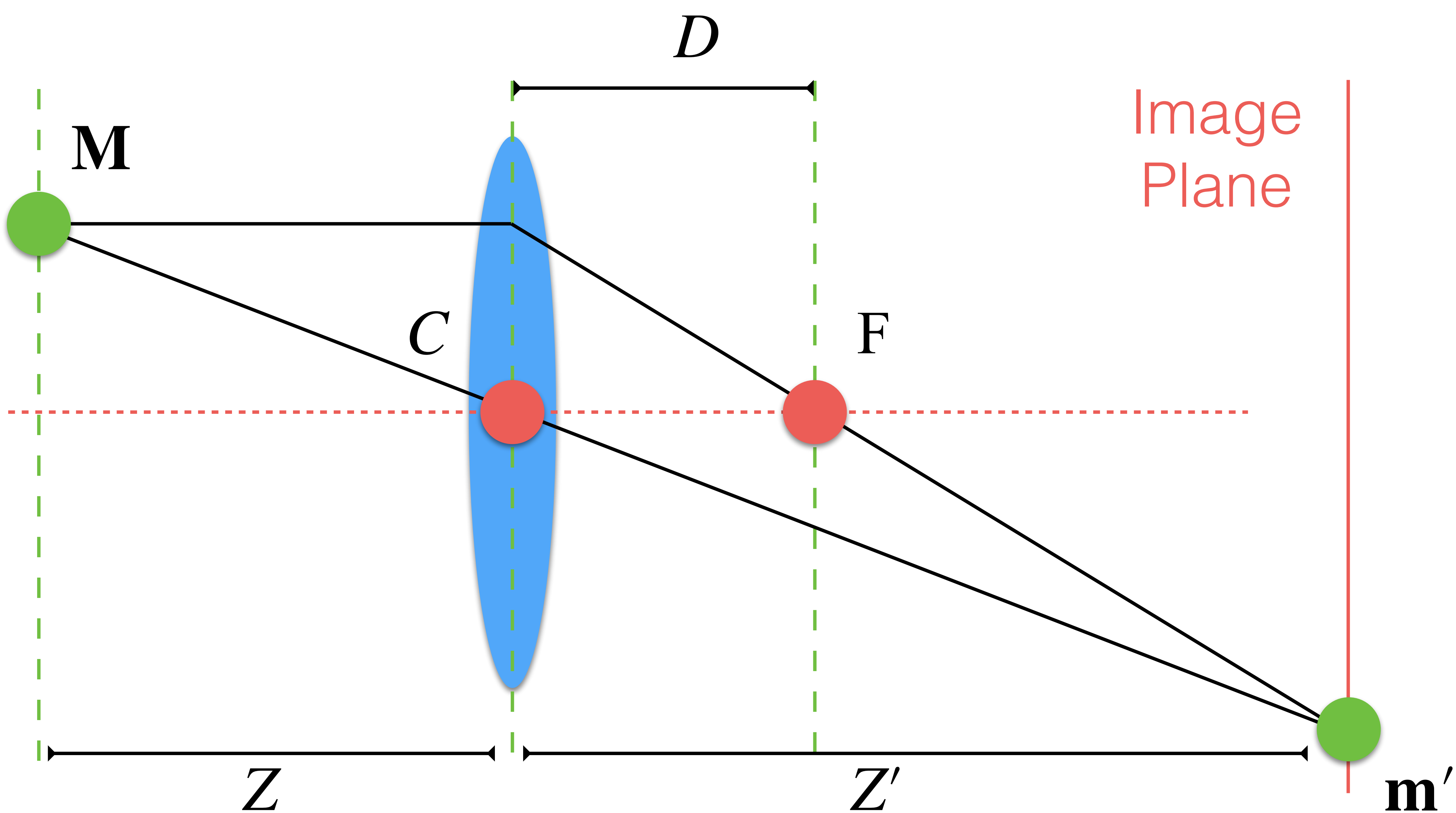
Extrinsic Matrix

Camera Model: Thin Lens



Camera Model: Thin Lens

$$\frac{1}{Z} + \frac{1}{Z'} = \frac{1}{D}$$



Camera Model: Thin Lens

- Typically, we model a thin lens system after projection:

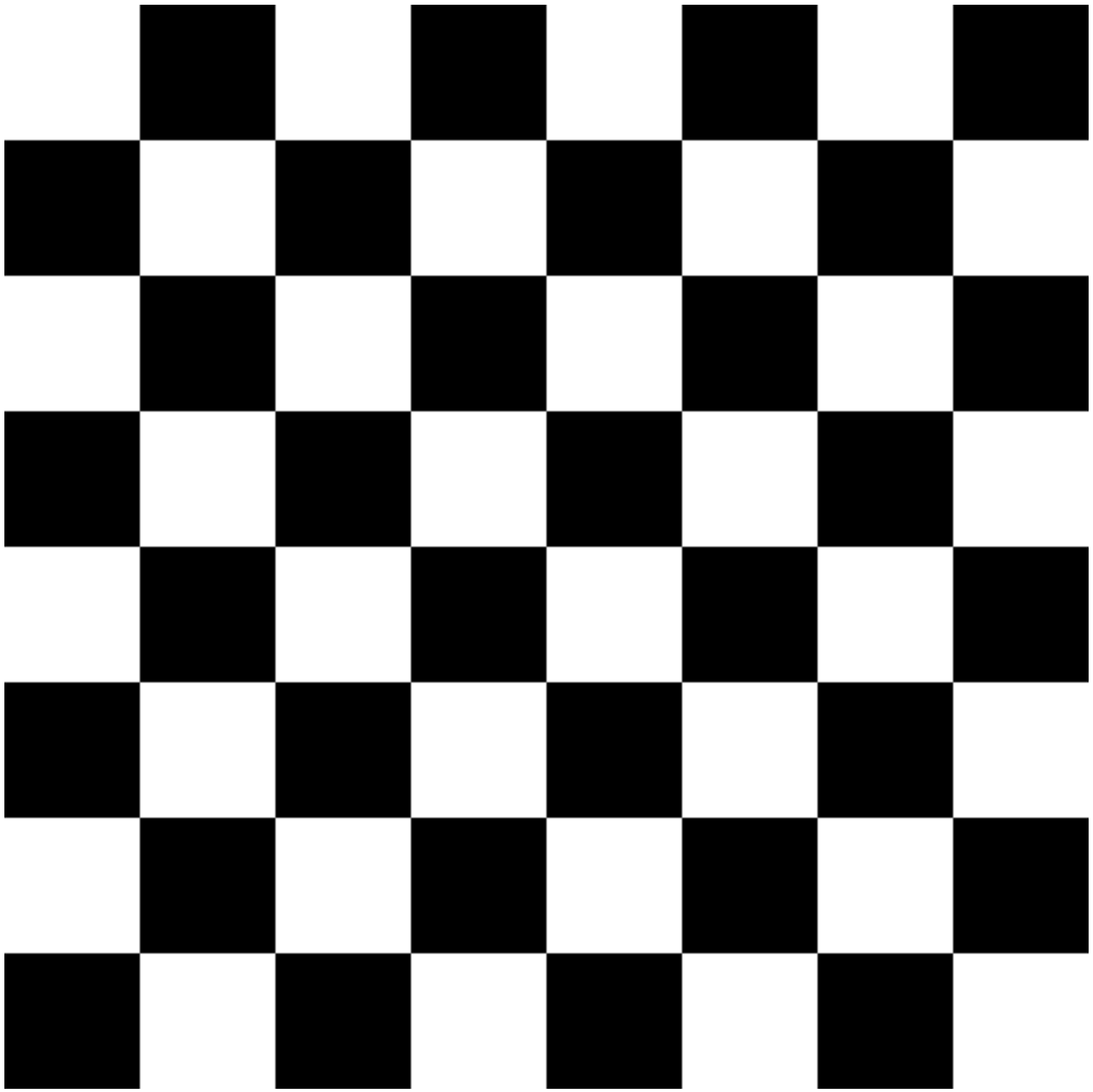
$$\mathbf{m}' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \begin{cases} u' = (u - u_0)(1 + k_1 r_d^2 + \ldots + k_n r_n^{2n}) + u_0 \\ v' = (v - v_0)(1 + k_1 r_d^2 + \ldots + k_n r_n^{2n}) + v_0 \end{cases}$$

$$\mathbf{m} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \sim P \cdot \mathbf{M}$$

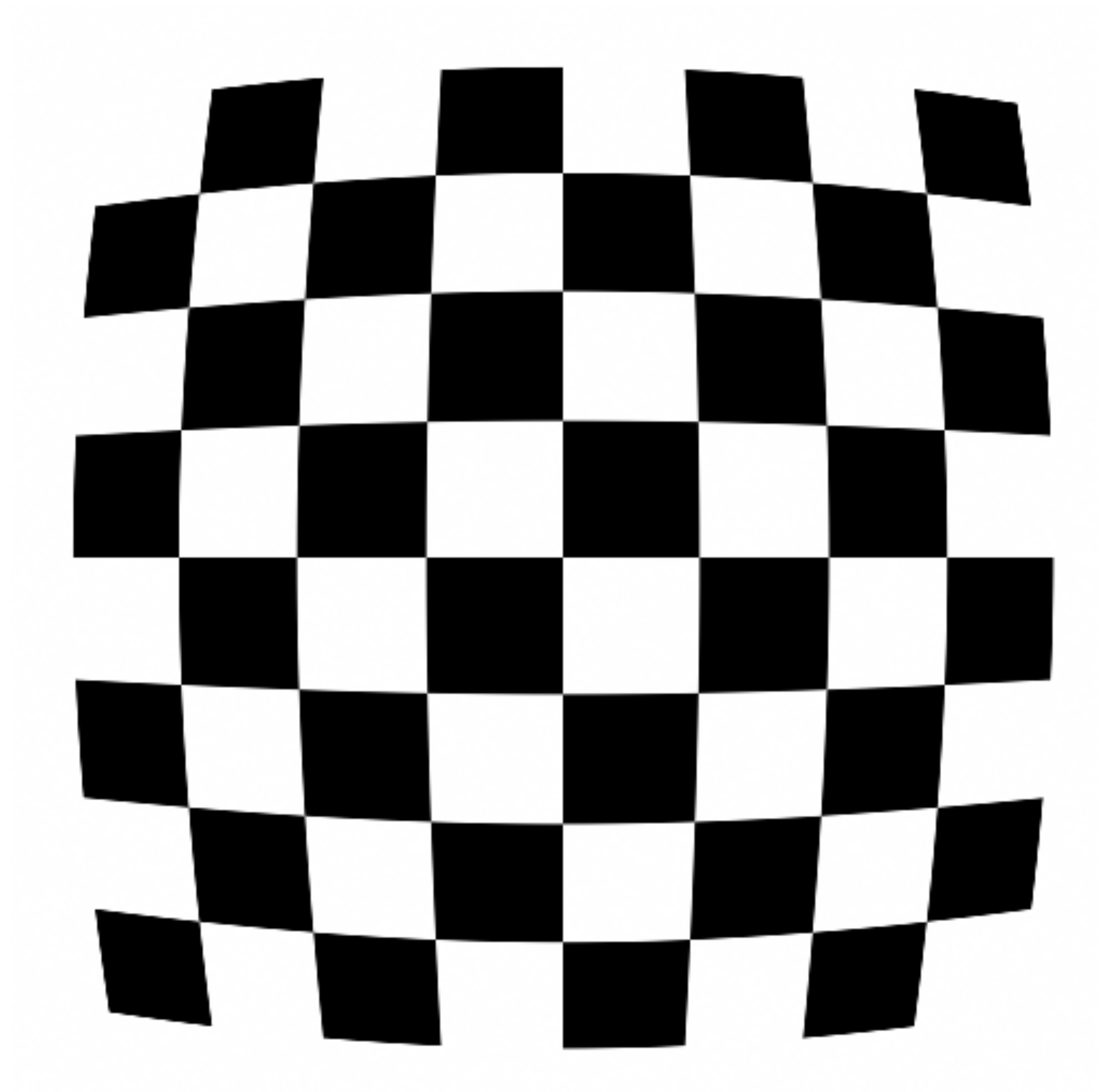
where n is usually set to 3, and r_d is defined as:

$$r_d^2 = \left(\frac{(u - u_o)^2}{\alpha_u^2} + \frac{(v - v_o)^2}{\alpha_v^2} \right)^2 \qquad \alpha_u = -f k_u \qquad \alpha_v = -f k_v$$

Camera Model: Thin Lens

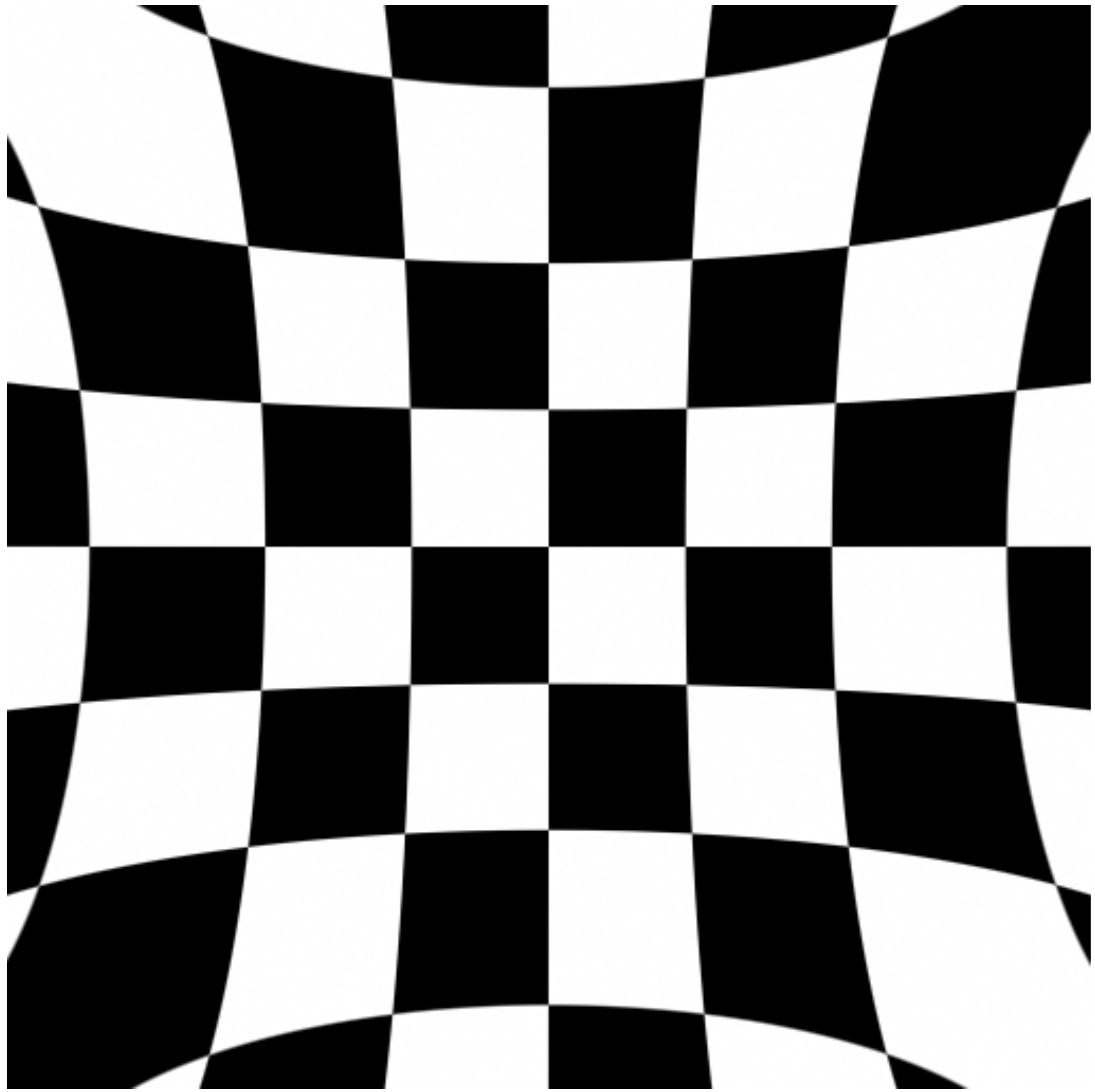


Camera Model: Thin Lens



Barrel distortion

Camera Model: Thin Lens



Pincushion distortion

Camera Pre-Calibration

Pre-Calibration: Why?

- In some cases, when we know the camera, it is useful to avoid intrinsics matrix estimation:
  - It is more precise.
  - We reduce computations.

Pre-Calibration: Parameters Estimation

- If we can have an "***estimation***" of K from camera parameters that are available in the camera specifications:

$$K = \begin{bmatrix} a & 0 & u_o \\ 0 & b & v_o \\ 0 & 0 & 1 \end{bmatrix}$$

- What do we need?
    - Focal length of the camera in mm (f), we can obtain it from the EXIF file of the JPEG/RAW file.
    - Resolution of the picture in pixels (w, h), we can find it in the manual of the camera or from the manufacturer specifications.
    - CCD/CMOS sensor size in mm (w_s, h_s), we can find it in the manual of the camera or from the manufacturer specifications.

Pre-Calibration: Parameters Estimation

- $a = f \cdot w / w_s$
- $b = f \cdot h / h_s$
- $u_0 = w/2$
- $v_0 = h/2$

Pre-Calibration: Parameters Estimation

- $a = f \cdot w / w_s$
- $b = f \cdot h / h_s$
- $u_0 = w/2$
- $v_0 = h/2$

Assuming it in the center!

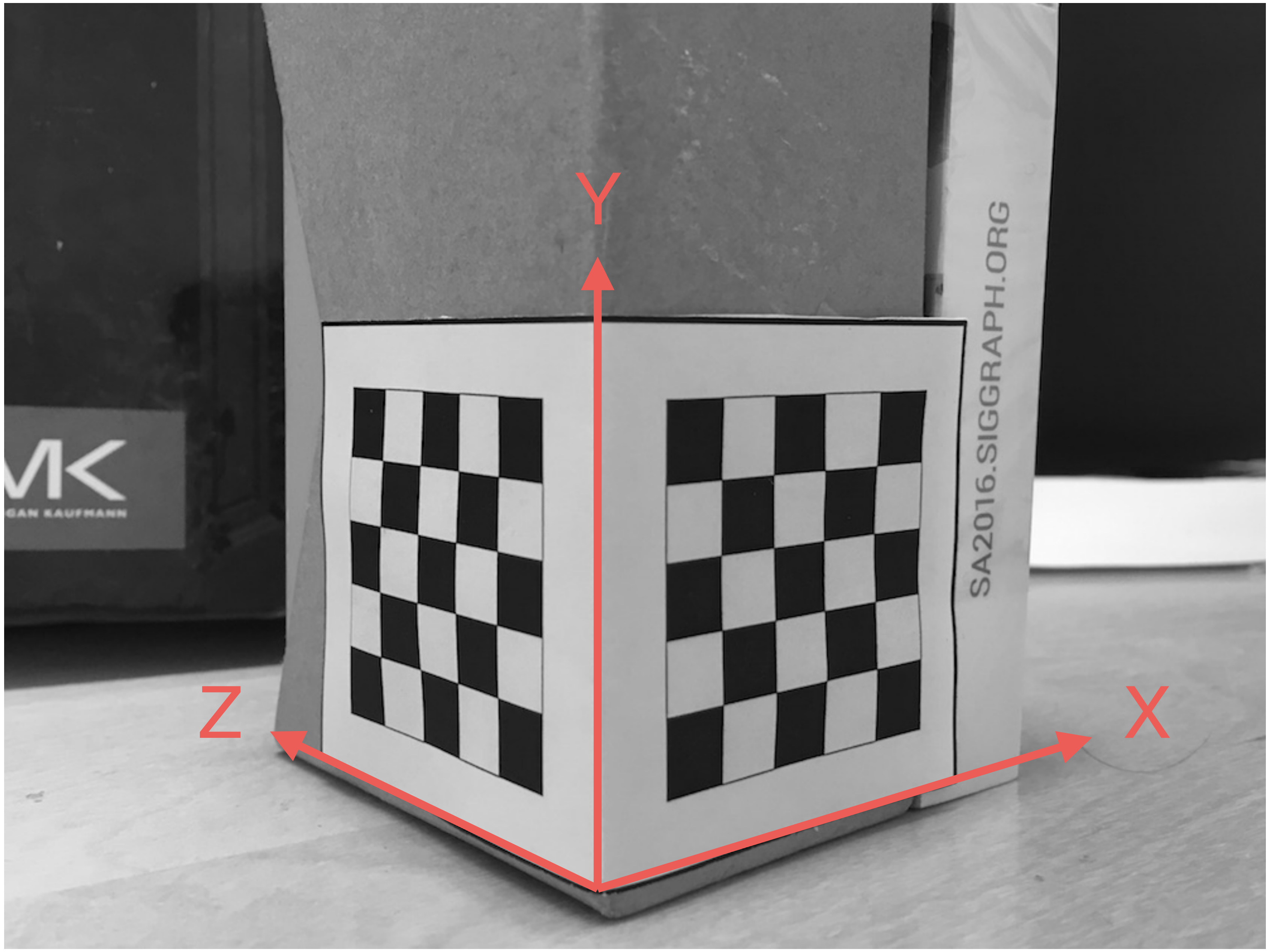
DLT:
Direct Linear Transform

DLT: Direct Linear Transform

- **Input**: a photograph of a non-coplanar calibration **object** (e.g., a checkerboard) with n 2D points (extracted manually or automatically) with known 3D coordinates (we know them because we built the calibration **object**!).

- **Output**: K of the camera. We can optionally recover $[R \mid \mathbf{t}]$.

DLT: Direct Linear Transform



DLT: Idea

$$\mathbf{m}_i = [u_i, v_i, 1]^\top \leftrightarrow \mathbf{M}_i = [x, y, z, 1]^\top$$

2D-3D matches

DLT: Idea

- At this point, if we get the projection equation back, we can notice that we know something:

$$P = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix} \qquad \begin{cases} u_i = \dfrac{\mathbf{p}_1^\top \cdot \mathbf{M}_i}{\mathbf{p}_3^\top \cdot \mathbf{M}_i} \\ \\ v_i = \dfrac{\mathbf{p}_2^\top \cdot \mathbf{M}_i}{\mathbf{p}_3^\top \cdot \mathbf{M}_i} \end{cases}$$

DLT: Idea

$$\begin{cases} \mathbf{p}_1^\top \cdot \mathbf{M}_i - u_i \mathbf{p}_3^\top \cdot \mathbf{M}_i = 0 \\ \mathbf{p}_2^\top \cdot \mathbf{M}_i - v_i \mathbf{p}_3^\top \cdot \mathbf{M}_i = 0 \end{cases}$$

$$\boxed{\mathbf{m}_i = [u_i, v_i, 1]^\top \leftrightarrow \mathbf{M}_i = [x, y, z, 1]^\top}$$

2D-3D matches

DLT: Linear System

- This leads to:

$$\begin{bmatrix} \mathbf{M}_i^\top & \mathbf{0} & -u_i \mathbf{M}_i^\top \\ \mathbf{0} & -\mathbf{M}_i^\top & v_i \mathbf{M}_i^\top \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{0}$$

- For each point, we need to stack this equations obtaining a matrix A.

- We obtain a $2n \times 12$ linear system to solve.

- The minimum number of points to solve is 6, but more points are required to have robust and stable solutions.

What's the problem with this method?

DLT: Direct Linear Transform

- DLT minimizes an algebraic error:
  - It does not have **geometric meaning**!!!
- Hang on, is it all wrong?
  - Nope, we can use it as input for a non-linear method.

DLT: Non-linear Refinement

- The non-linear refinement minimizes (at least squares) the distance between 2D points of the image ($\mathbf{m}_i$) and projected 3D points ($\mathbf{M}_i$):

$$\arg\min_{P} \sum_{i=1}^{n} \left(\frac{\mathbf{p}_1^\top \cdot \mathbf{M}_i}{\mathbf{p}_3^\top \cdot \mathbf{M}_i} - u_i \right)^2 + \left(\frac{\mathbf{p}_2^\top \cdot \mathbf{M}_i}{\mathbf{p}_3^\top \cdot \mathbf{M}_i} - v_i \right)^2$$

- Different methods for solving it such as Gradient Descent (*we need gradients*), Nelder-Mead's method (MATLAB's **fminsearch**), etc.

Now we have a nice matrix P…

DLT: Direct Linear Transform

- Let's recap:
  - K has to be upper-triangular. $K = \begin{bmatrix} -fk_u & 0 & u_0 \\ 0 & -fk_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}$
  - R is orthogonal.
  - $P = K[R|\mathbf{t}] = [K \cdot R|K \cdot \mathbf{t}] = [P'|\mathbf{p}_4]$

DLT: Direct Linear Transform

- QR decomposition of a matrix A:

$$A = O \cdot T$$

  where:

  - O is orthogonal.
  - T is **upper-triangular**.

- In our case, we have:

$$P' = K \cdot R \rightarrow (P')^{-1} = R^{-1} \cdot K^{-1}$$

DLT: Direct Linear Transform

- We apply the QR decomposition to P' as:

$$[(P')^{-1}]_{\mathrm{QR}} = O \cdot T$$

- In our case, we have:

$$R = O^{-1} = O^{\top} \qquad K = T^{-1}$$

- We compute $\mathbf{t}$ as:

$$\mathbf{t} = K^{-1} \cdot \mathbf{p}_4 = T \cdot \mathbf{p}_4$$

and what’s about the radial distortion?

Estimating Radial Distortion

- Let's start with simple radial distortion; i.e., only a coefficient:

$$\begin{cases} u' = (u - u_0) \cdot (1 + k_1 r_d^2) + u_0 \\ v' = (v - v_0) \cdot (1 + k_1 r_d^2) + v_0 \end{cases}$$

$$r_d^2 = \left(\frac{(u - u_0)}{\alpha_u} \right)^2 + \left(\frac{(v - v_0)}{\alpha_v} \right)^2 \quad \alpha_u = -f \cdot k_u \quad \alpha_u = -f \cdot k_v$$

- Can we solve it?

Estimating Radial Distortion

- We have only one unknown, which is linear; i.e., k_1:

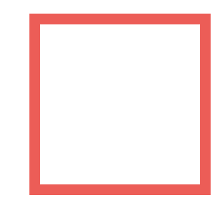
$$\begin{cases} k_1 = \dfrac{u' - u}{(u - u_0)r_d^2} \\ k_1 = \dfrac{v' - v}{(v - v_0)r_d^2} \end{cases}$$

- In theory, ***a single point is enough***, but it is better to use more points to get a more robust solution.

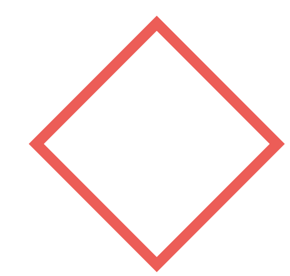
Homography

2D Transformations

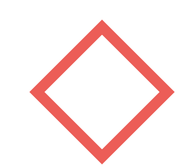
- We can have different type of transformation (defined by a matrix) of 2D points:
  - Translation (2 degree of freedom [DoF]):

    - It preserves orientation.
  - Rigid/Euclidian (3 DoF); translation, and rotation:

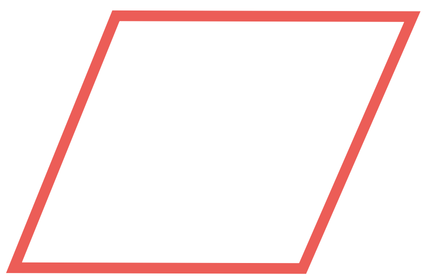
    - It preserves lengths.
  - Similarity (4DoF); translation, rotation, and scaling:

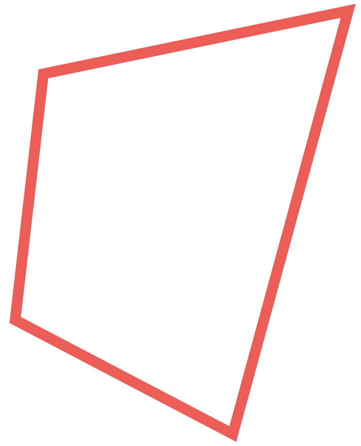
    - It preserves angles.

2D Transformations

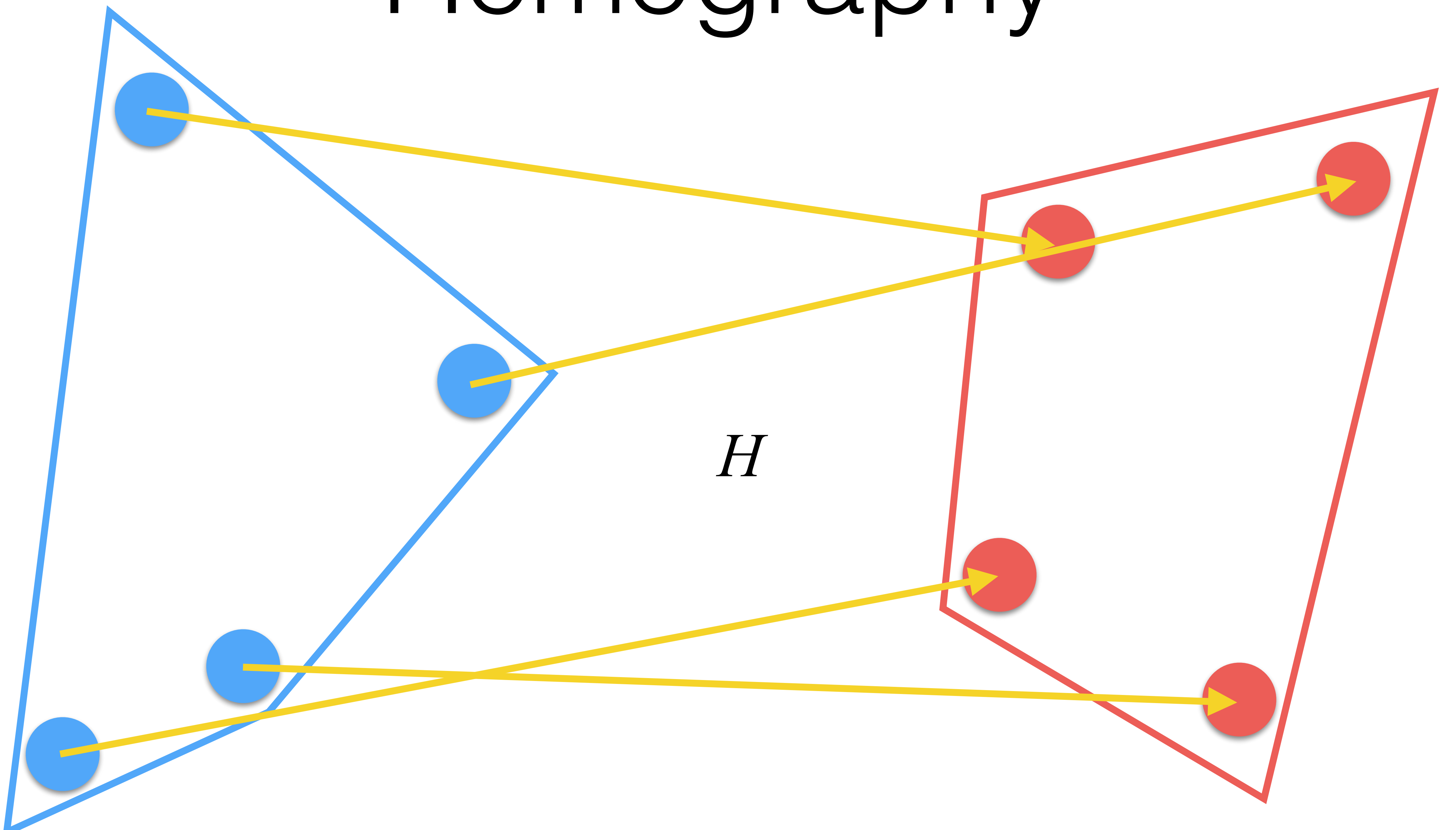
- Affine (6 degree of freedom [DoF]):
  - It reserves parallelism.



- Projective (8 DoF):
  - It preserves straight lines.



2D Transformations: Homography



$$\mathbf{m}_a = \begin{bmatrix} x_a \\ y_a \\ 1 \end{bmatrix} \qquad \mathbf{m}_b = \begin{bmatrix} x_b \\ y_b \\ 1 \end{bmatrix}$$

2D Transformations: Homography

- Homography is defined as

$$\hat{\mathbf{m}}_b = H \cdot \mathbf{m}_a \quad \rightarrow \quad \mathbf{m}_b = \begin{bmatrix} x_b \\ y_b \\ 1 \end{bmatrix} = \hat{\mathbf{m}}_b \frac{1}{\hat{m}_{b,3}}$$

- This is typically expressed as

$$\mathbf{m}_b \sim H \cdot \mathbf{m}_a$$

where H is a 3×3 non-singular matrix with 8 DoF.

Homography Estimation

$$\mathbf{m}_b \sim H \cdot \mathbf{m}_a \qquad \mathbf{m}_a = \begin{bmatrix} x_a \\ y_a \\ 1 \end{bmatrix} \qquad \mathbf{m}_b = \begin{bmatrix} x_b \\ y_b \\ 1 \end{bmatrix}$$

$$\mathbf{m}_b = \begin{cases} x_b = \dfrac{h_{11}x_a + h_{12}y_a + h_{13}}{h_{31}x_a + h_{32}y_a + h_{33}} \\[2ex] y_b = \dfrac{h_{21}x_a + h_{22}y_a + h_{23}}{h_{31}x_a + h_{32}y_a + h_{33}} \end{cases}$$

Homography Estimation

$$\begin{cases} x_b(h_{31}x_a + h_{32}y_a + h_{33}) - h_{11}x_a + h_{12}y_a + h_{13} = 0 \\ y_b(h_{31}x_a + h_{32}y_a + h_{33}) - h_{21}x_a + h_{22}y_a + h_{23} = 0 \end{cases}$$

Stacking multiple equations;
one for each match (at least 5!)

$$A \cdot \mathbf{vec}(H) = \mathbf{0}$$

Homography Estimation

- Again, we have minimized an algebraic error!!
- Technically speaking, we should run a non-linear optimization:

$$\arg\min_{H} \sum_{i=1}^{m} \left(x_{b,i} - \frac{\mathbf{h}_1^\top \cdot \mathbf{m}_{a,i}}{\mathbf{h}_3^\top \cdot \mathbf{m}_{a,i}} \right)^2 + \left(y_{b,i} - \frac{\mathbf{h}_2^\top \cdot \mathbf{m}_{a,i}}{\mathbf{h}_3^\top \cdot \mathbf{m}_{a,i}} \right)^2$$

where $H = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 \end{bmatrix}$, and m is the number of matched point.

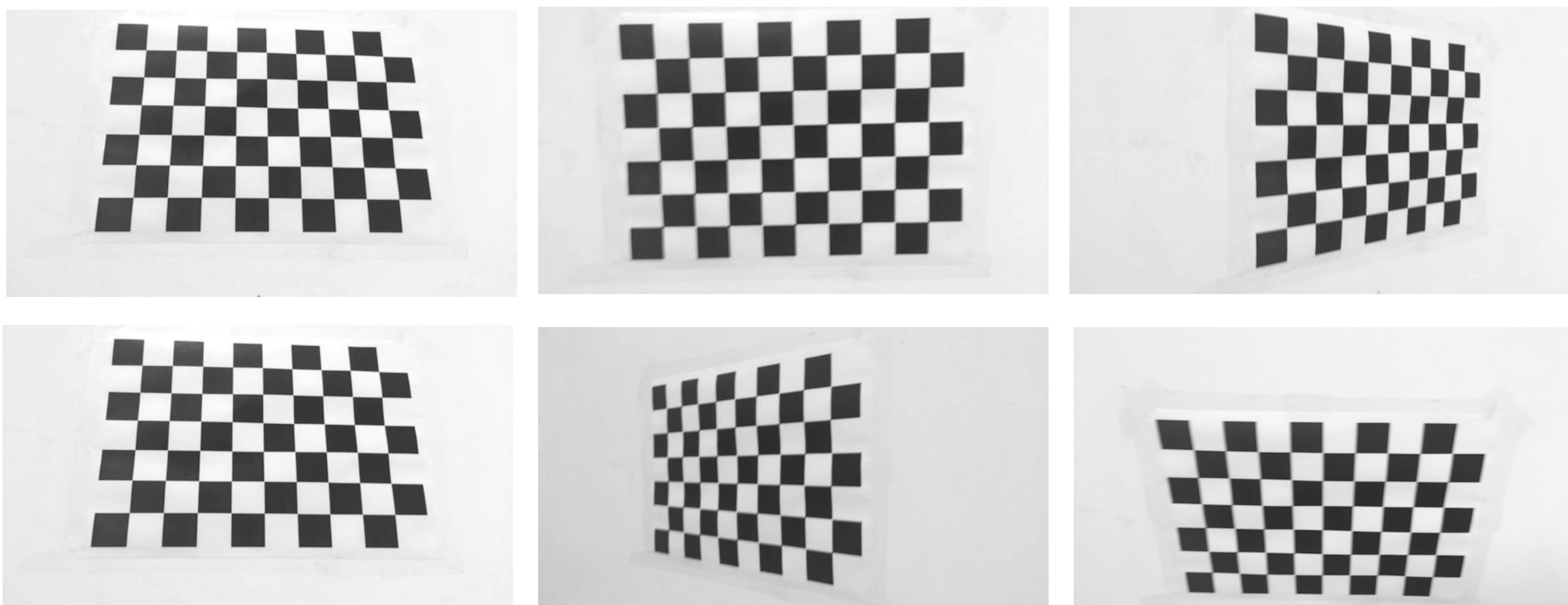
Zhang’s Method

Zhang's Method

- **Input**: a set of n photographs capturing a checkerboard (fully visible) or other patterns. From these, we have to extract m points (i.e., all corners of a checker!) in each photograph.

- **Output**: K. We can optionally compute $G = [R\,|\,\mathbf{t}]$ for each photograph.

Zhang's Method



A set of input images

Zhang's Method

$$K = \begin{bmatrix} -fk_u & 0 & u_0 \\ 0 & -fk_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Zhang's Method

$$K = \begin{bmatrix} \alpha & c & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: c is a function of the angle between the u-axis and v-axis in the image plane.

Zhang's Method

$$K = \begin{bmatrix} \alpha & \boxed{c} & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: c is a function of the angle between the u-axis and v-axis in the image plane.

Zhang's Method

- **Assumption:**
  - We have a set of photographs of a plane so Z is equal 0.
  - So we have 3D points defined as

$$\mathbf{M} = \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix}$$

Zhang's Method

- This means that we have: $\mathbf{m} \sim P \cdot \mathbf{M}$

$$P \cdot \mathbf{M} = K \cdot [R|\mathbf{t}] \cdot \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} =$$

$$= K \cdot [\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 | \mathbf{t}] \cdot \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} =$$

$$= K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

Zhang's Method

$$= K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Zhang's Method

$$= K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Zhang's Method

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This is a homography!

Zhang's Method

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Zhang's Method

$$= K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is a homography!

$$H = K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}]$$

Zhang's Method

$$= K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is a homography!

$$H = K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}]$$

Zhang's Method

$$= K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is a homography!

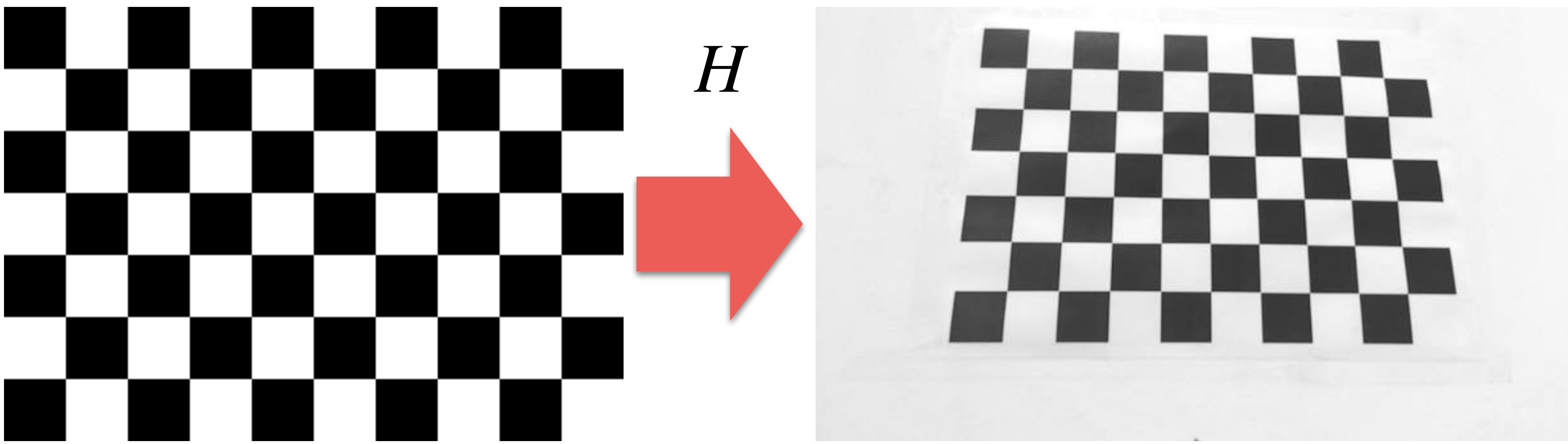
$$H = K \cdot [\mathbf{r}_1 \mathbf{r}_2 | \mathbf{t}]$$

$$H = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 \end{bmatrix}$$

Zhang's Method

- What to do?
- For each photograph:
  - We compute the homography H between photographed checkerboard corners and its model.

Zhang's Method



Model

Photograph

Zhang's Method

- At this point, starting from H for each photograph, we need to compute K, R, and $\mathbf{t}$.
- $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are orthonormal, so we have:

1. $$\mathbf{h}_1^\top K^{-\top} K^{-1} \mathbf{h}_2 = 0$$

2. $$\mathbf{h}_1^\top K^{-\top} K^{-1} \mathbf{h}_1 = \mathbf{h}_2^\top K^{-\top} K^{-1} \mathbf{h}_2$$

Zhang's Method

- Note that all K parameters can be compressed into:

$$B = K^{-\top} K^{-1} = \begin{bmatrix} \frac{1}{\alpha^2} & -\frac{c}{\alpha^2\beta} & \frac{cv_0 - u_0\beta}{\alpha^2\beta} \\ -\frac{c}{\alpha^2\beta} & \frac{c^2}{\alpha^2\beta^2} + \frac{1}{\beta^2} & -\frac{c(cv_0 - u_0\beta)}{\alpha^2\beta^2} - \frac{v_0}{\beta^2} \\ \frac{cv_0 - u_0\beta}{\alpha^2\beta} & -\frac{c(cv_0 - u_0\beta)}{\alpha^2\beta^2} - \frac{v_0}{\beta^2} & \frac{(cv_0 - u_0\beta)^2}{\alpha^2\beta^2} + \frac{v_0^2}{\beta^2} + 1 \end{bmatrix}$$

- B is symmetric → defined only by six values:

$$\mathbf{vec}(B) = [b_{11}, b_{12}, b_{22}, b_{13}, b_{23}, b_{33}]^\top$$

Zhang's Method

- Given that:

$$\mathbf{h}_1^\top K^{-\top} K^{-1} \mathbf{h}_1 = \mathbf{h}_2^\top K^{-\top} K^{-1} \mathbf{h}_2$$

- We have so:

$$\mathbf{h}_i^\top \cdot B \cdot \mathbf{h}_j = \mathbf{v}_{ij}^\top \cdot \mathbf{vec}(B)$$

- where:

$$H = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 \end{bmatrix}$$

$$\mathbf{h}_i = \begin{bmatrix} h_{i1} & h_{i2} & h_{13} \end{bmatrix}^\top$$

$$\mathbf{v}_{ij} = \begin{bmatrix} h_{i1}h_{j1} \\ h_{i1}h_{j2} + h_{i2}h_{j1} \\ h_{i2}h_{j2} \\ h_{i3}h_{j1} + h_{i1}h_{j3} \\ h_{i3}h_{j2} + h_{i2}h_{j3} \\ h_{i3}h_{j3} \end{bmatrix}$$

Zhang's Method

- Given that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are orthonormal, and that:

$$\mathbf{h}_1^\top K^{-\top} K^{-1} \mathbf{h}_2 = 0$$

- We obtain:

$$\begin{bmatrix} \mathbf{v}_{12}^\top \\ (\mathbf{v}_{11} - \mathbf{v}_{12})^\top \end{bmatrix} \cdot \mathbf{vec}(B) = \mathbf{0}$$

Zhang's Method

- If n images of the model plane are observed, we obtain the following by stacking n of such equations:

$$\begin{bmatrix} \mathbf{v}_{12}^{\top} \\ (\mathbf{v}_{11} - \mathbf{v}_{12})^{\top} \end{bmatrix} \cdot \mathbf{vec}(B) = \mathbf{0}$$

- This leads to:

$$V \cdot \mathbf{vec}(B) = \mathbf{0}$$

- V is $2n \times 6$ matrix, so we need $n > 2$.

Zhang's Method

- At this point, we can compute elements of K as

$$u_0 = (b_{12}b_{13} - b_{11}b_{23})/(b_{11}, b_{22} - b_{12}^2)$$

$$\lambda = b_33 - (b_{13}^2 + v_0(b_{12}b_{13} - b_{11} - b_{23}))/b_{11}$$

$$\alpha = \sqrt{\lambda - b_{11}}$$

$$\beta = \sqrt{\lambda b_{11}/(b_{11}b_{22} -_{12}^2)}$$

$$c = -b_{12}\alpha^2\beta/\lambda$$

$$u_0 = cv_0/\alpha - b_{13}\alpha^2/\lambda$$

Zhang's Method

- Furthermore, we can extract the pose as

$$\mathbf{r}_1 = \lambda \cdot K^{-1}\mathbf{h}_1$$
$$\mathbf{r}_2 = \lambda \cdot K^{-1}\mathbf{h}_2$$
$$\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$$
$$\mathbf{t} = \lambda K^{-1}\mathbf{h}_3$$

Zhang's Method: Non-Linear Refinement

- So far, we have obtained a solution through minimizing an algebraic distance that is not physically meaningful!

- From that solution, we can use a non-linear method for minimizing the following error:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left\|\mathbf{m}_{i,j}-\tilde{\mathbf{m}}(K, R_i, \mathbf{t}_i, \mathbf{M}_j)\right\|^2$$

- where n is the number of photographs, and m is the number of matched points.

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This function projects $\mathbf{M}_j$ points (3D) given K and $G_i = [R_i | \mathbf{t}_i]$.

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Zhang's Method: Optical Distortion

- What's about the parameters for modeling the radial distortion?

$$\begin{bmatrix} (u-u_0)r_d^2 & (u-u_0)r_d^4 \\ (v-v_0)r_d^2 & (v-v_0)r_d^4 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} u'-u \\ v'-v \end{bmatrix}$$

$$r_d^2 = \left(\frac{(u-u_0)}{\alpha_u}\right)^2 + \left(\frac{(v-v_0)}{\alpha_v}\right)^2 \quad \alpha_u = -f \cdot k_u \quad \alpha_u = -f \cdot k_v$$

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$$D \cdot \mathbf{k} = \mathbf{d}$$

$$\mathbf{k} = (D^\top \cdot D)^{-1} \cdot D^\top \cdot \mathbf{d}$$

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- As before, first algebraic solution, and then a non-linear solution.

- We extend the previous non-linear model to include optical distortion:

- $$\sum_{i=1}^{n}\sum_{j=1}^{m}\left\|\mathbf{m}_{i,j} - \tilde{\mathbf{m}}(K, R_i, \mathbf{t}_i, \mathbf{M}_j, \mathbf{k})\right\|^2$$

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that’s all folks!