Monte Carlo

Applications

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Applications
Introduction

- Monte-Carlo methods and integration can be applied in several fields:
  - Deep Learning
  - Imaging
  - Computer Graphics
  - Finance
  - Chemistry
  - Physics

A 2D Problem: Image Filtering

The Bilateral Filter

Introduction

- The bilateral filter is a non-linear filter for images and videos.
- It works in spatial domain and intensity/range domain of an image/video.

- Basically, it is an adaptive linear filter:
  - It behaves as a linear filter in flat regions;
  - At strong edges (step-edge), filtering is “limited”.

The Bilateral Filter

Introduction

Spatial Function

Range Function

$$BF[I](\mathbf{x}, f_s, g_r) = \frac{1}{K(\mathbf{x}, f_s, g_r)} \sum_{\mathbf{y} \in \Omega(\mathbf{x})} I(\mathbf{y}) f_s(\|\mathbf{x} - \mathbf{y}\|) g_r(\|I(\mathbf{y}) - I(\mathbf{x})\|),$$

$$K[I](\mathbf{x}, f_s, g_r) = \sum_{\mathbf{y} \in \Omega(\mathbf{x})} f_s(\|\mathbf{x} - \mathbf{y}\|) g_r(\|I(\mathbf{y}) - I(\mathbf{x})\|),$$

The Bilateral Filter

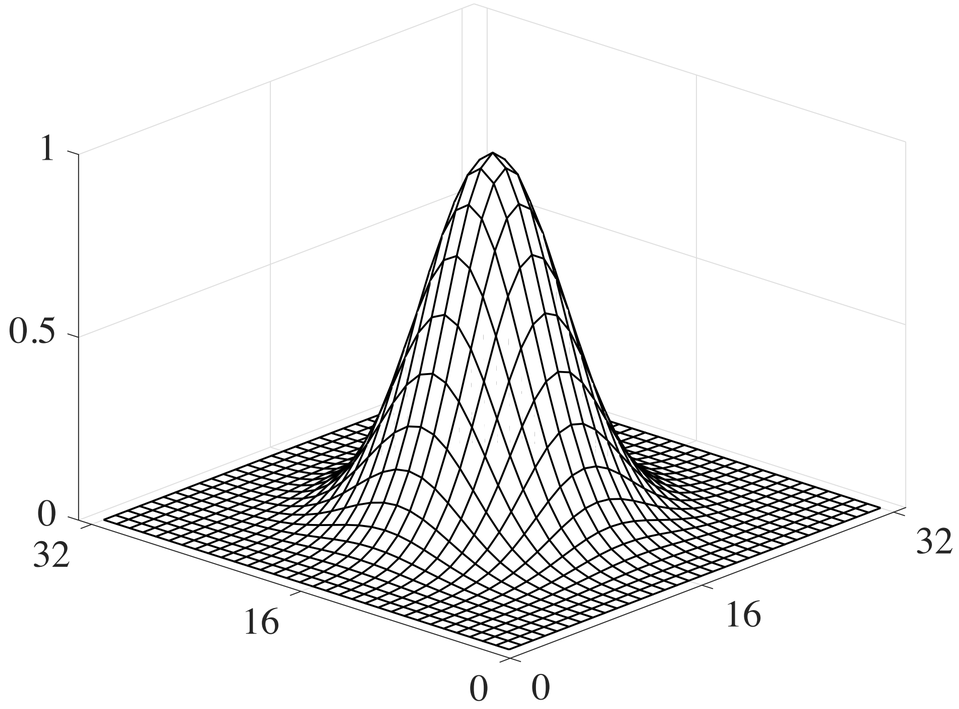
Introduction

- f_s (Spatial function): a Gaussian function
- g_r (Range function): a Gaussian function
- How large is the kernel?
  - If the spatial function is a Gaussian:

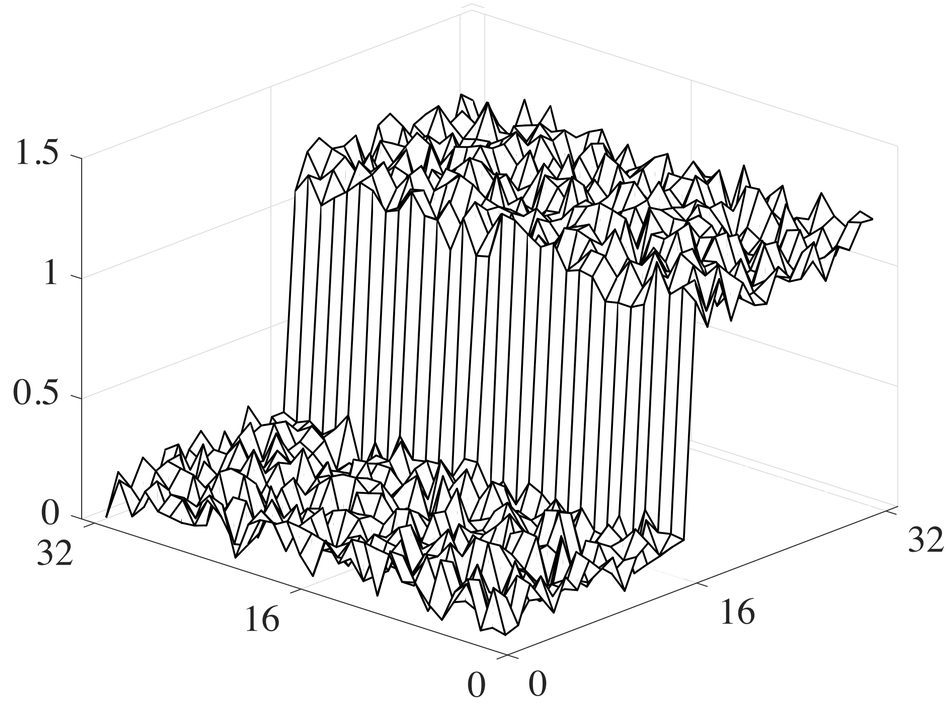
$$N = M = \frac{5}{2}\sigma_s \, .$$

The Bilateral Filter

Example



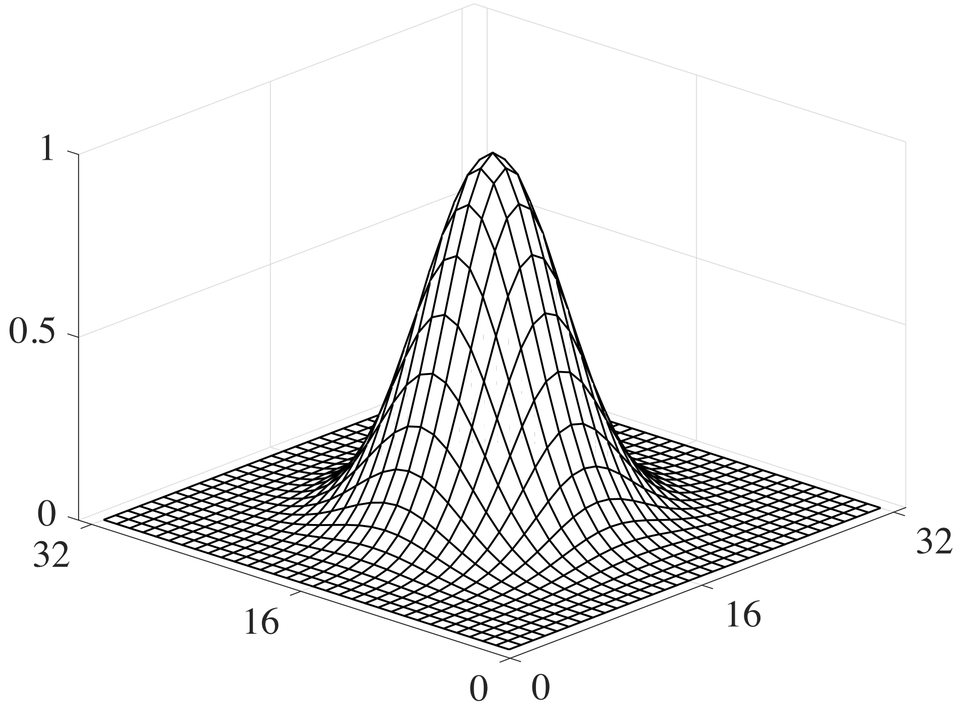
Kernel



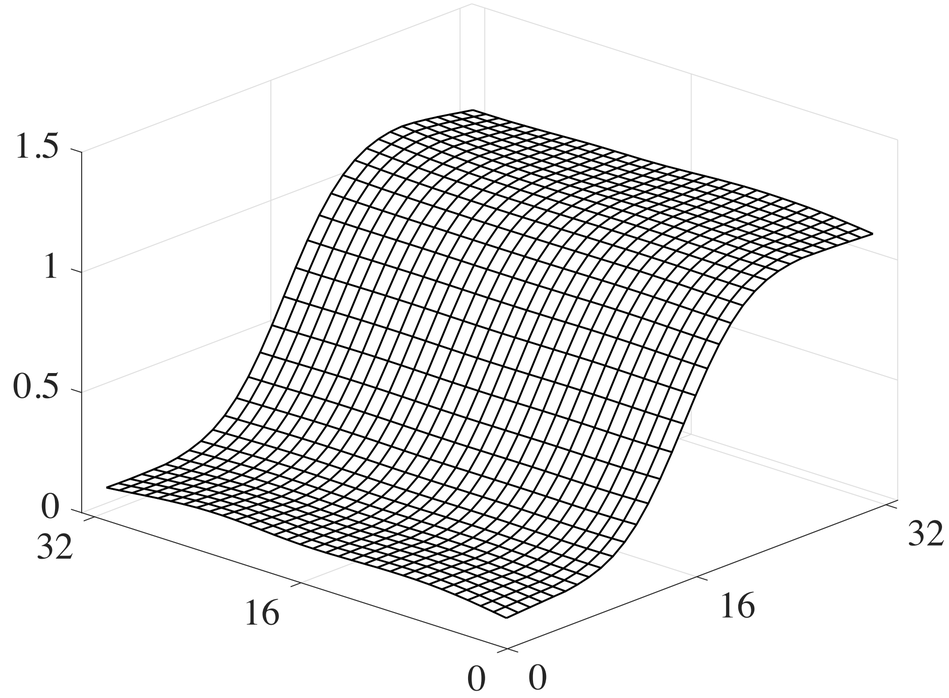
Image

The Bilateral Filter

Example



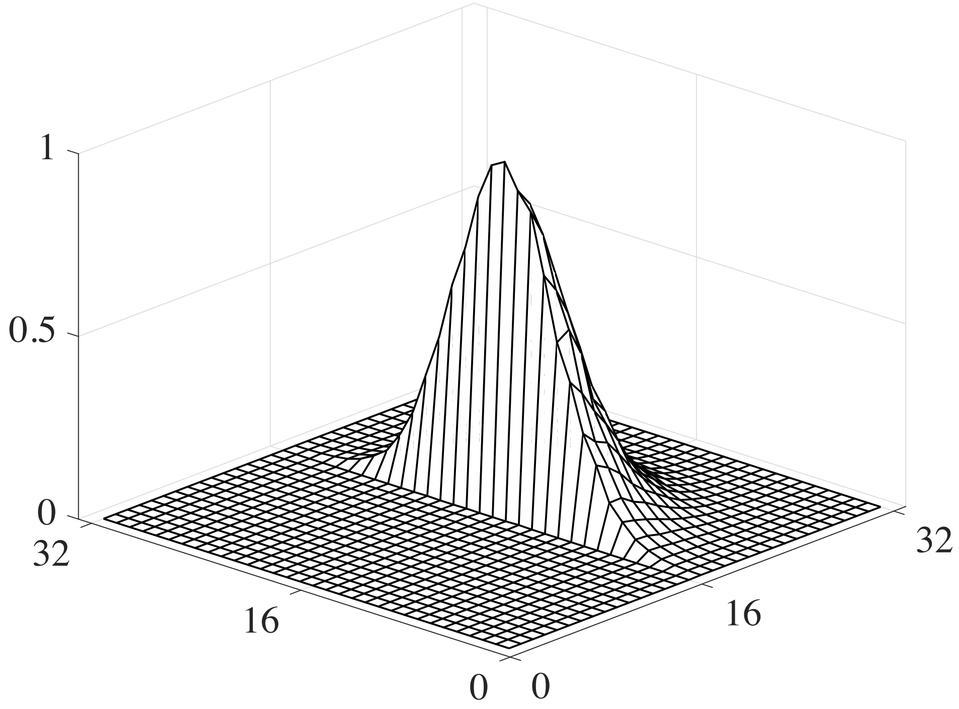
Kernel



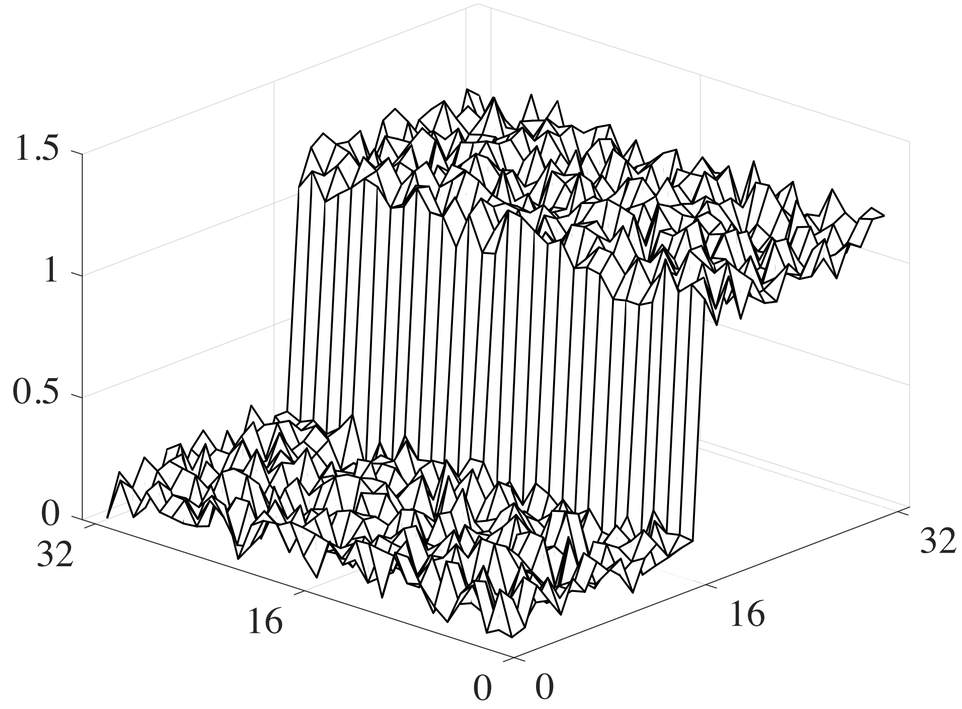
Image

The Bilateral Filter

Example



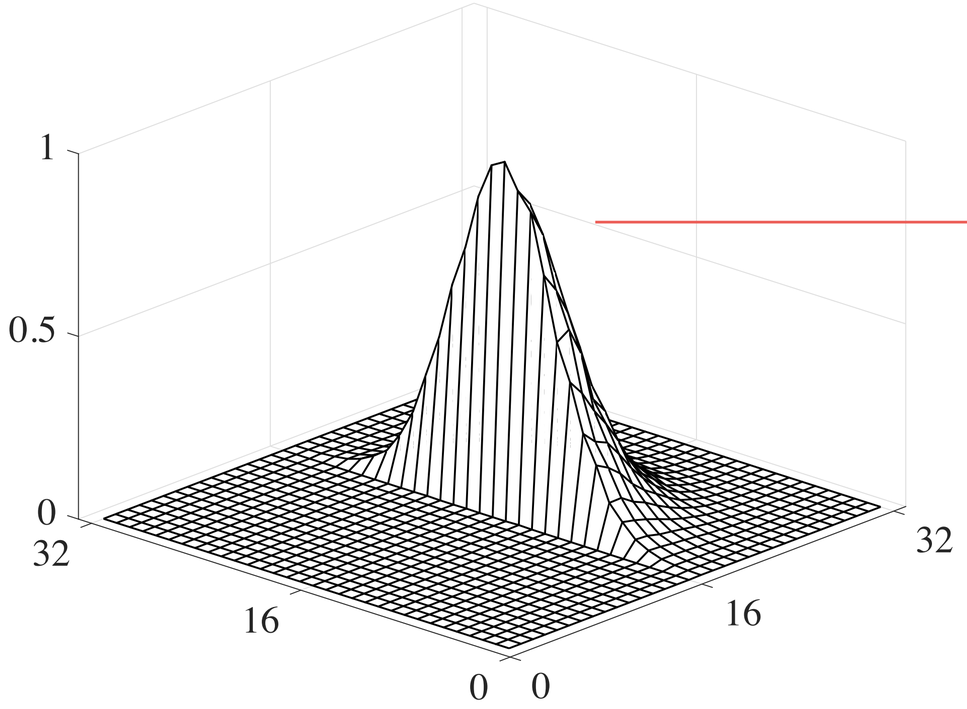
Kernel
(change for each pixel!!)



Image

The Bilateral Filter

Example



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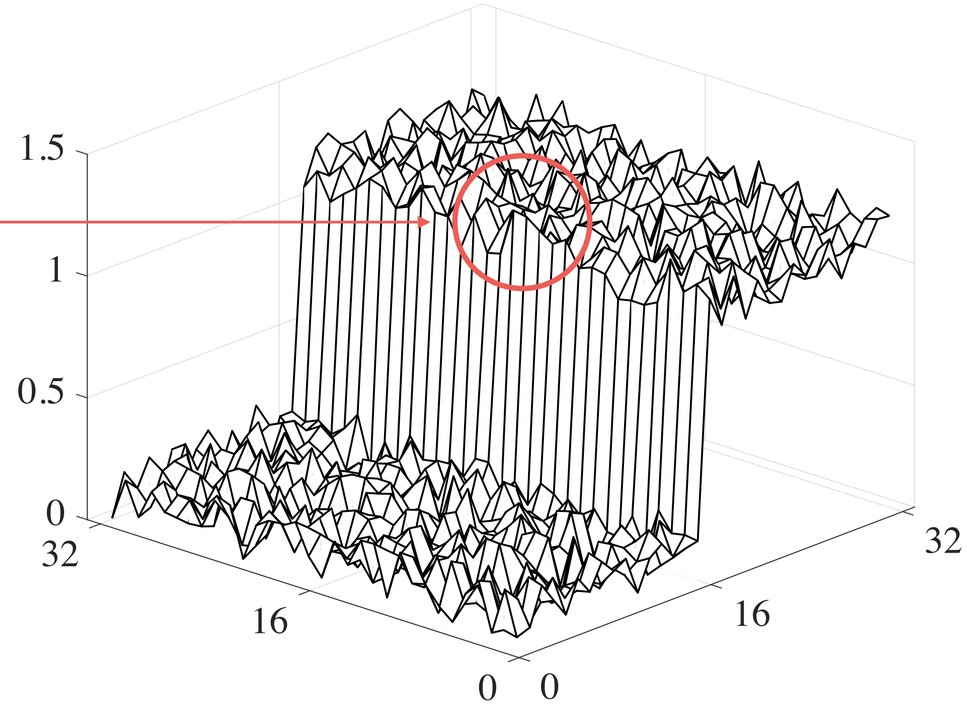
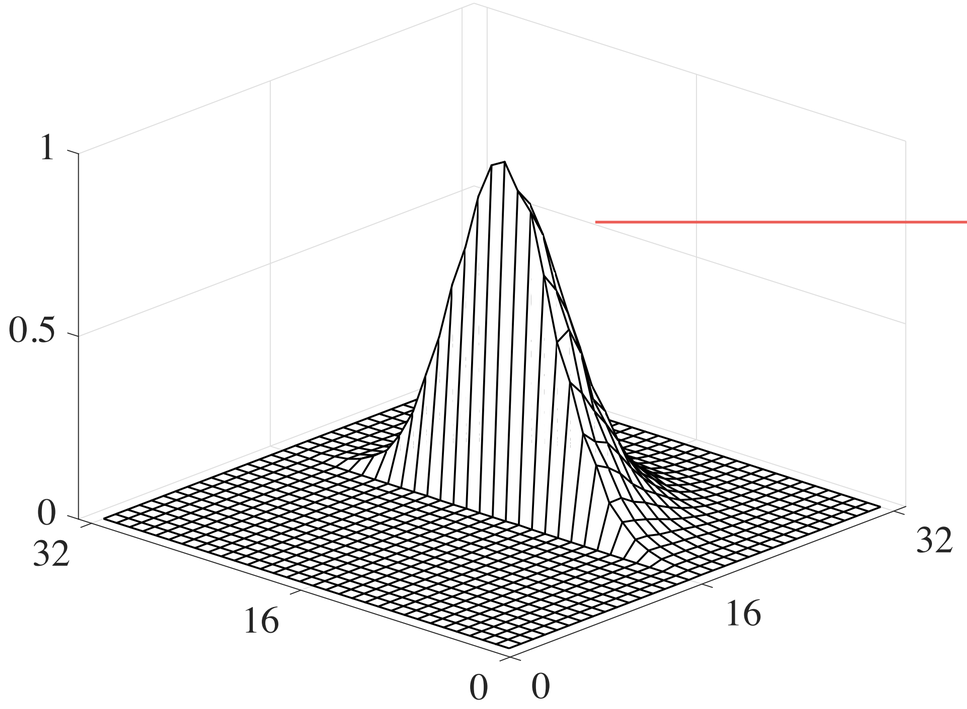


Image

The Bilateral Filter

Example



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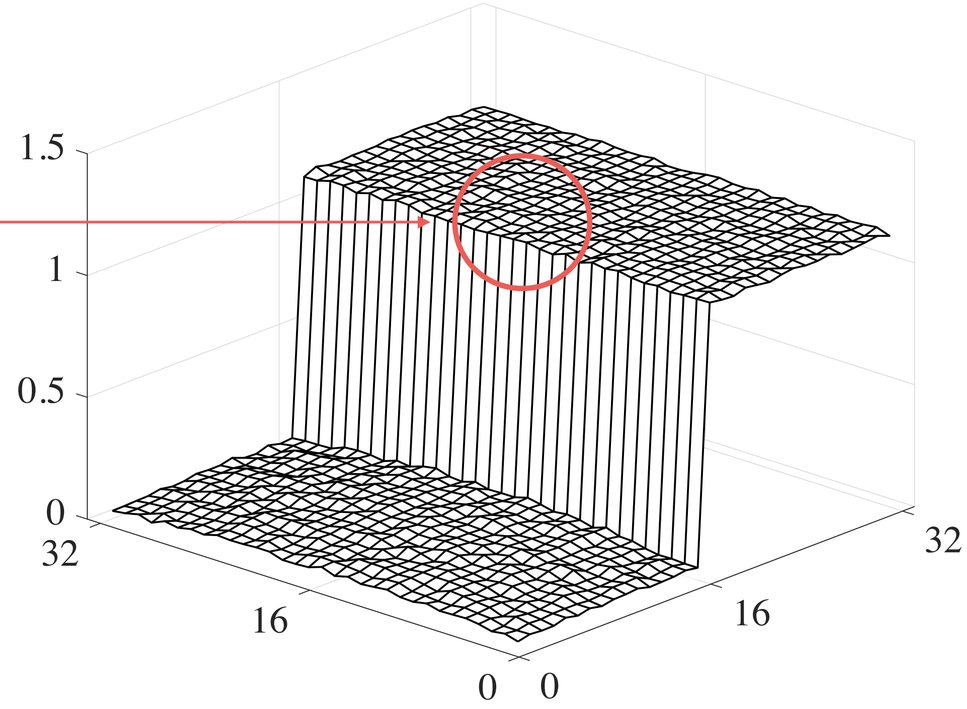


Image

The Bilateral Filter

Example

The Bilateral Filter

Example

The Bilateral Filter

Example

The Bilateral Filter

Example

The Bilateral Filter

Computational Complexity

- The main problem of the filter is its high computational complexity for real-time applications:

$$\mathcal{O}(nk^2),$$

  where n is the number of pixels of an image/video, and k is the size.

- Compared to a Gaussian filter:
  - Not separable;
  - No Fourier domain.

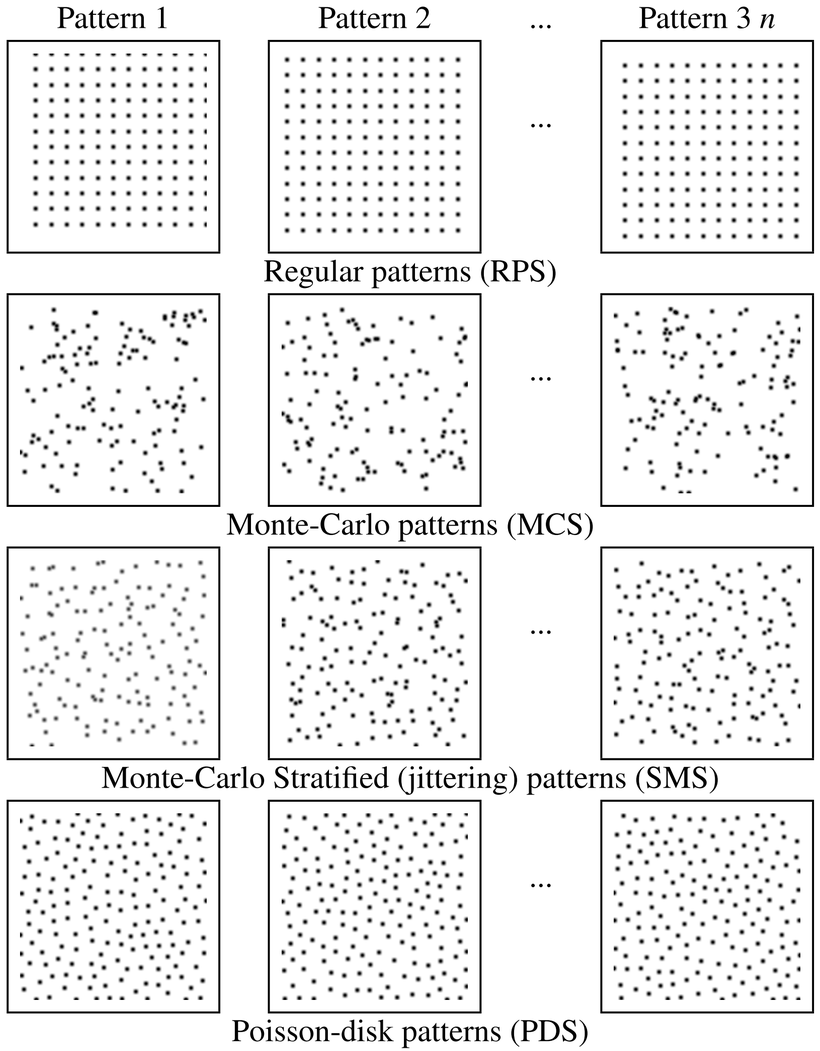
The Bilateral Filter

Monte-Carlo

- In this case, we can solve with Monte-Carlo!
- Basic idea:
  - We draw sample according the spatial Gaussian:
    - Box-Muller method.
  - We limit the number of samples to k or ck; with $c < k$ a constant.

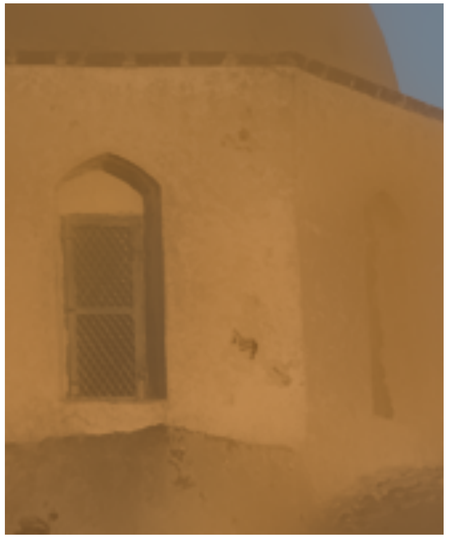
The Bilateral Filter

Sampling Strategies



The Bilateral Filter

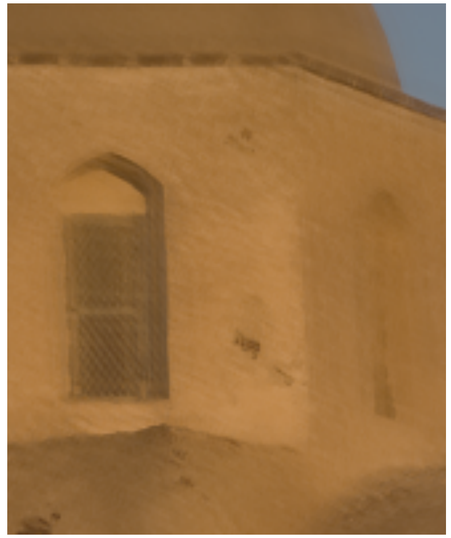
Sampling Strategies



Original

Poisson-Disk

Regular Grid

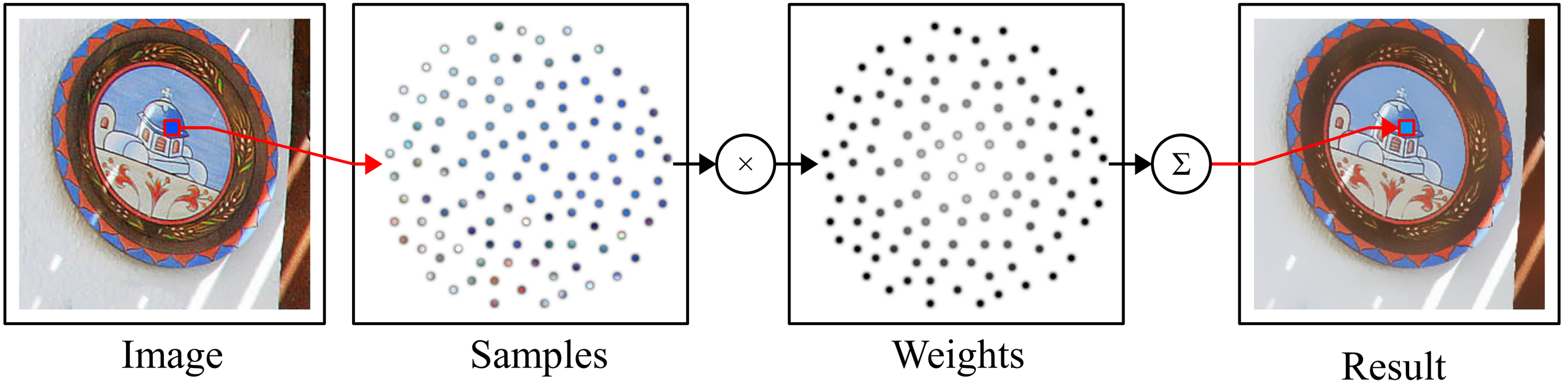


Monte-Carlo

Stratified
Monte-Carlo

The Bilateral Filter

Sampling Strategies



A Recursive Problem: Rendering

Path-Tracing

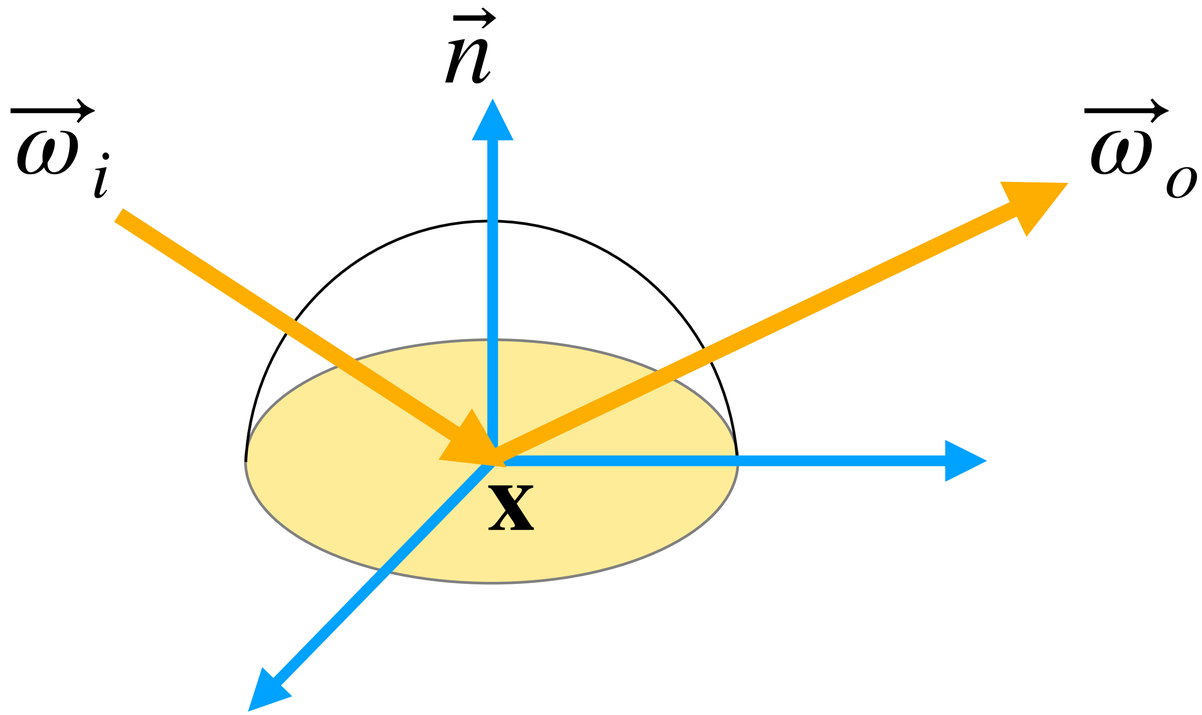
Introduction

- A classic problem in Computer Graphics is given:
  - Camera;
  - 3D Geometry;
  - Light sources’ description;
  - Materials’ description.
- To compute the color of each pixel in the image plane of our plane by simulating the light transport in a physically based manner.

Path-Tracing

The Rendering Equation

$$L_o(\mathbf{x}, \vec{\omega}_o, \lambda) = L_e(\mathbf{x}, \vec{\omega}_o, \lambda) + \int_\Omega f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_o, \lambda) L_i(\mathbf{x}, \vec{\omega}_i, \lambda) \, |\vec{n} \cdot \vec{\omega}_i| \, d\omega_i$$



Path-Tracing

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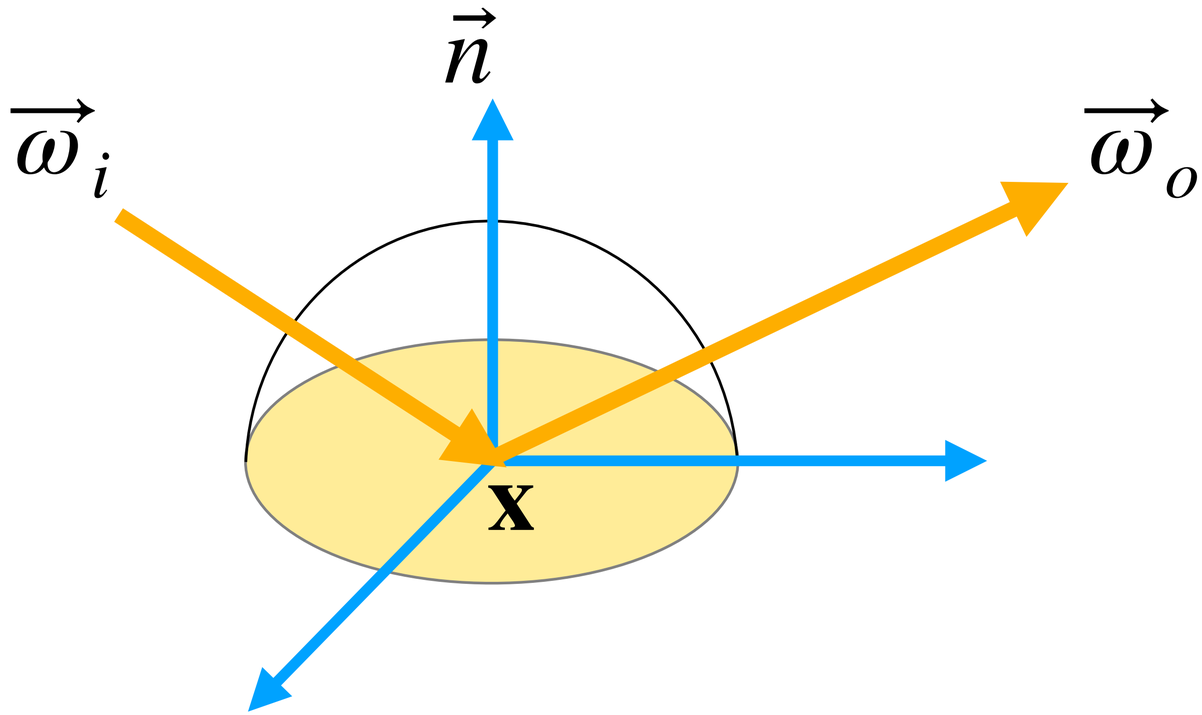

Path-Tracing

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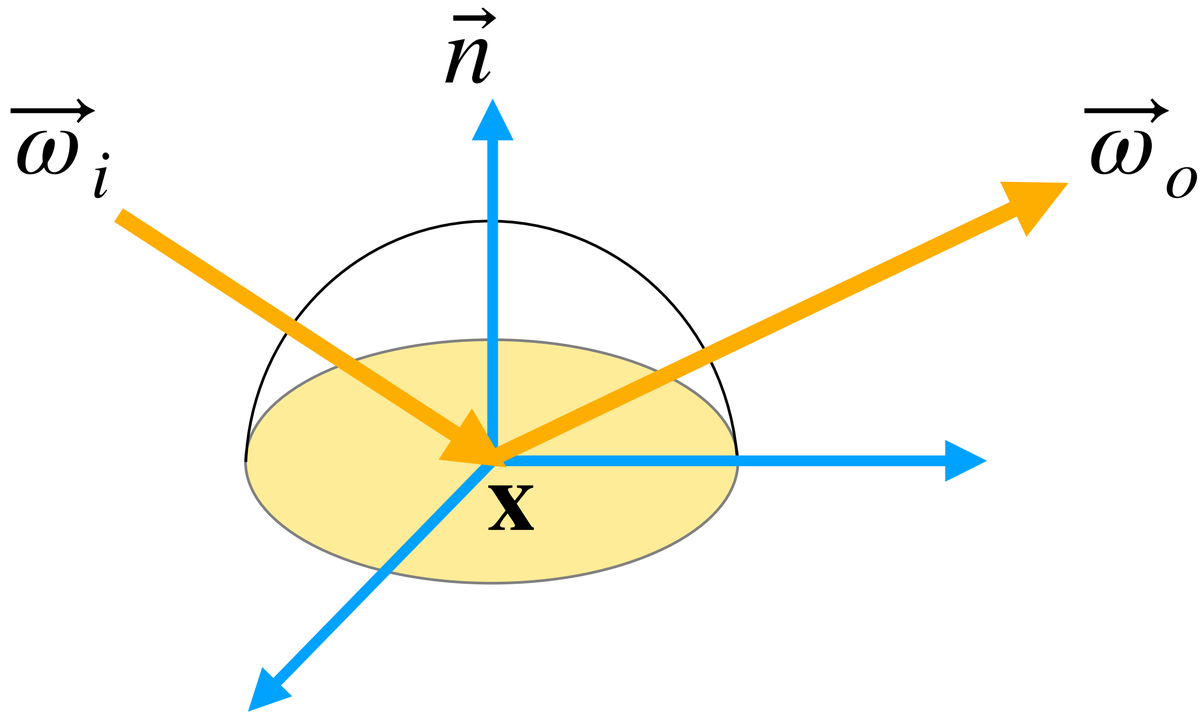

Path-Tracing

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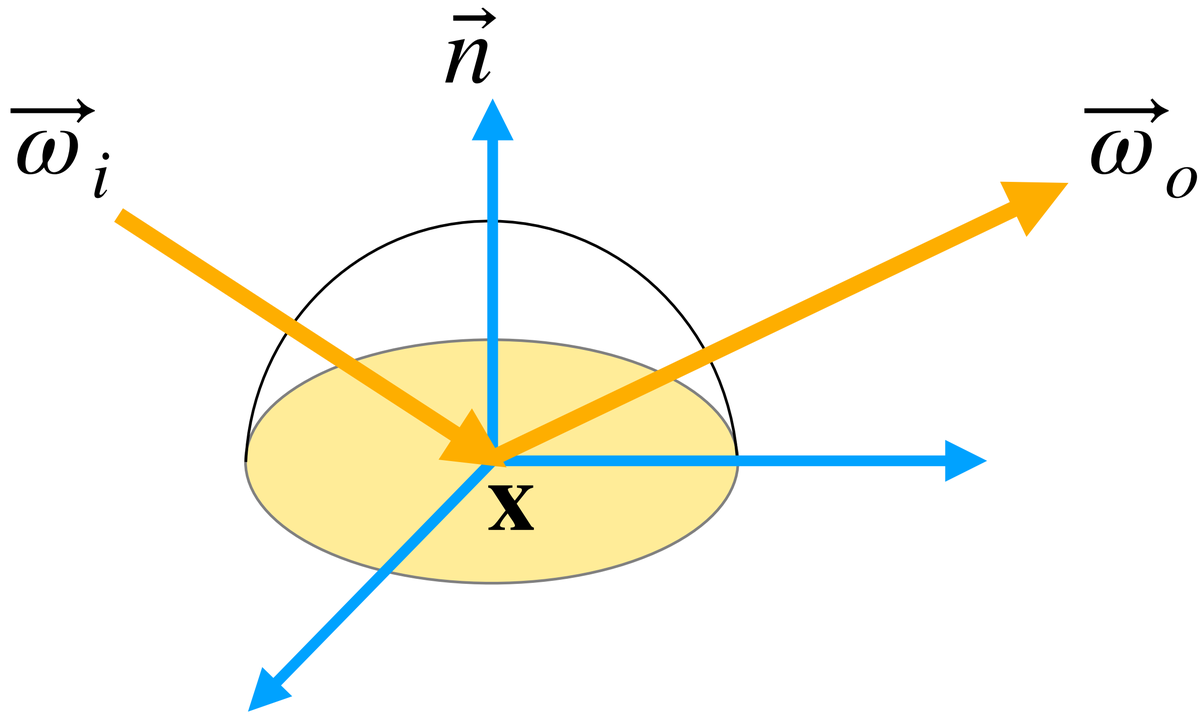

Path-Tracing

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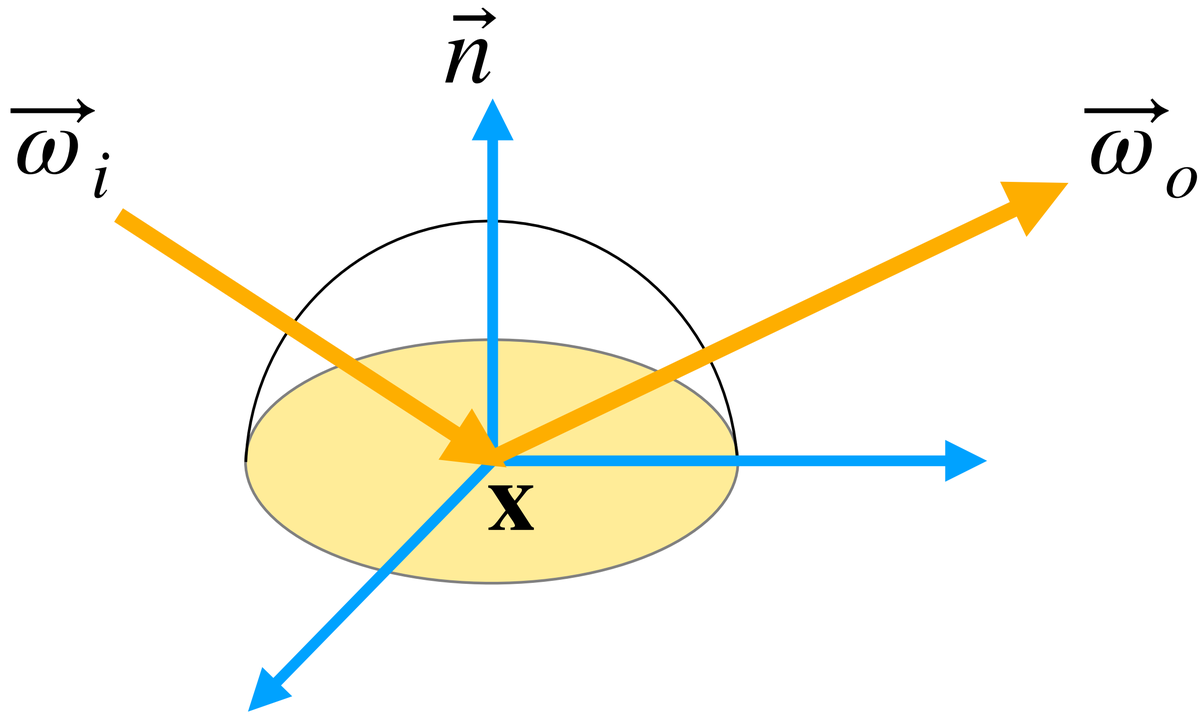
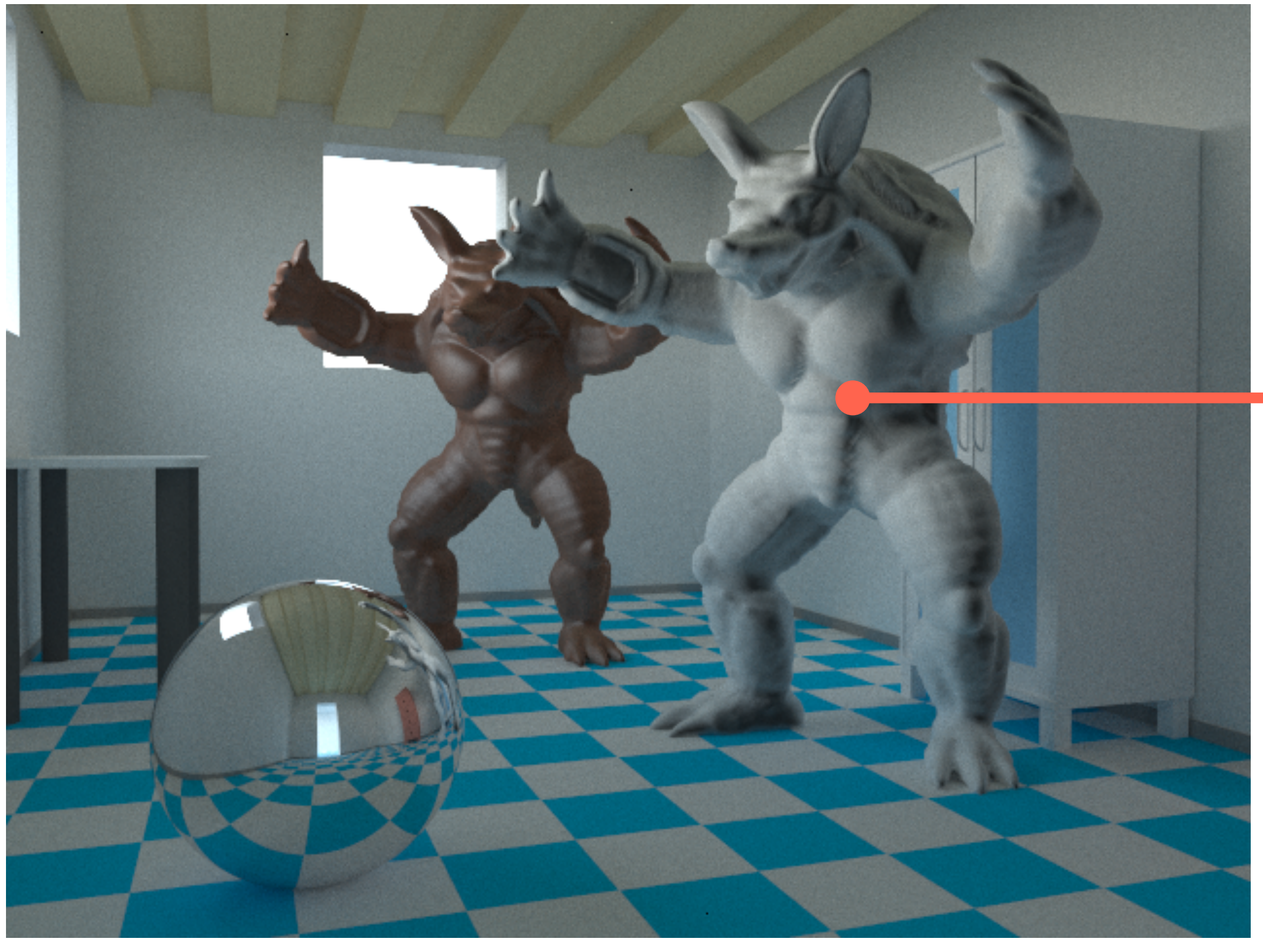
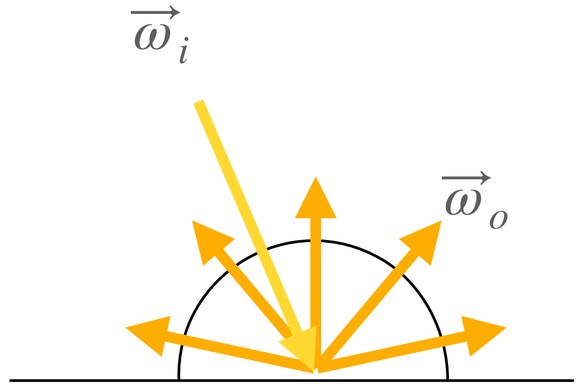


Path-Tracing

The Rendering Equation: BRDF

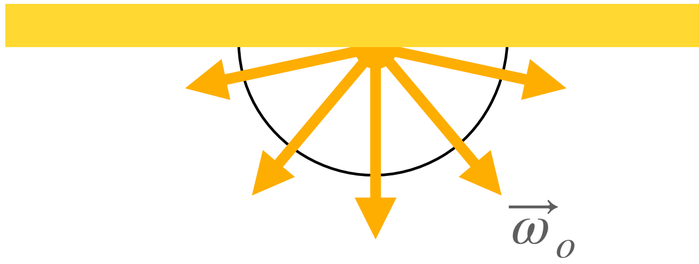




$$f_r(\mathbf{x}), \vec{\omega}_i, \vec{\omega}_o, \lambda) = \frac{\rho_\lambda}{\pi}$$

Path-Tracing

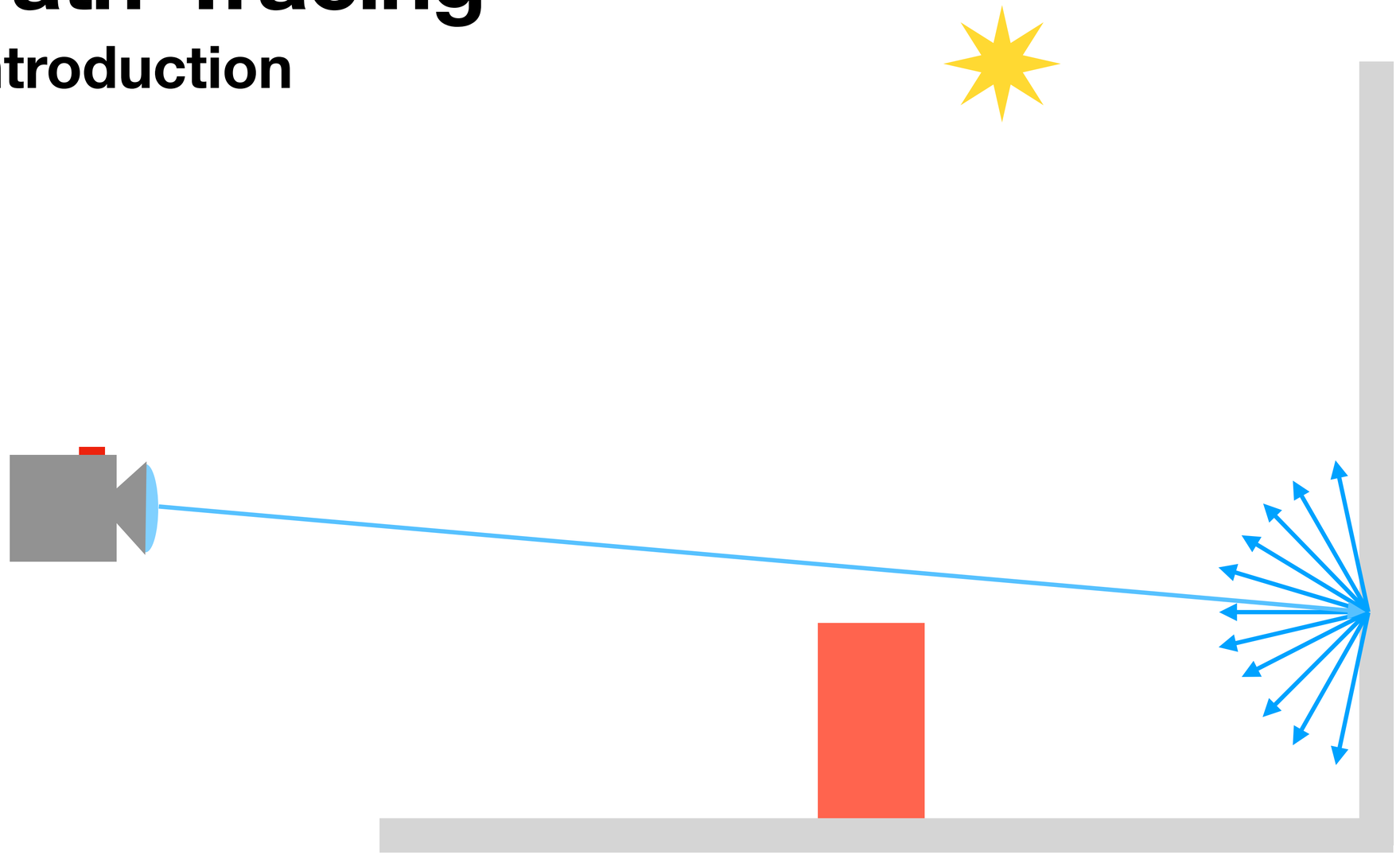
The Rendering Equation: Light Sources



$$I_e(\mathbf{x}, \vec{\omega}_o) = \frac{\Phi_\lambda}{\pi A}$$

Path-Tracing

Introduction



$$L_o(\mathbf{x}, \vec{\omega}_o, \lambda) = L_e(\mathbf{x}, \vec{\omega}_o, \lambda) + \int_\Omega f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_o) L_i(\mathbf{x}, \vec{\omega}_i, \lambda) \, |\vec{n} \cdot \vec{\omega}_i| \, d\omega_i$$

Path-Tracing

Introduction

- In a deterministic way, we should shoot n rays at each bounce for each location:

$$\sum_k n^k,$$

  and this highly impractical.

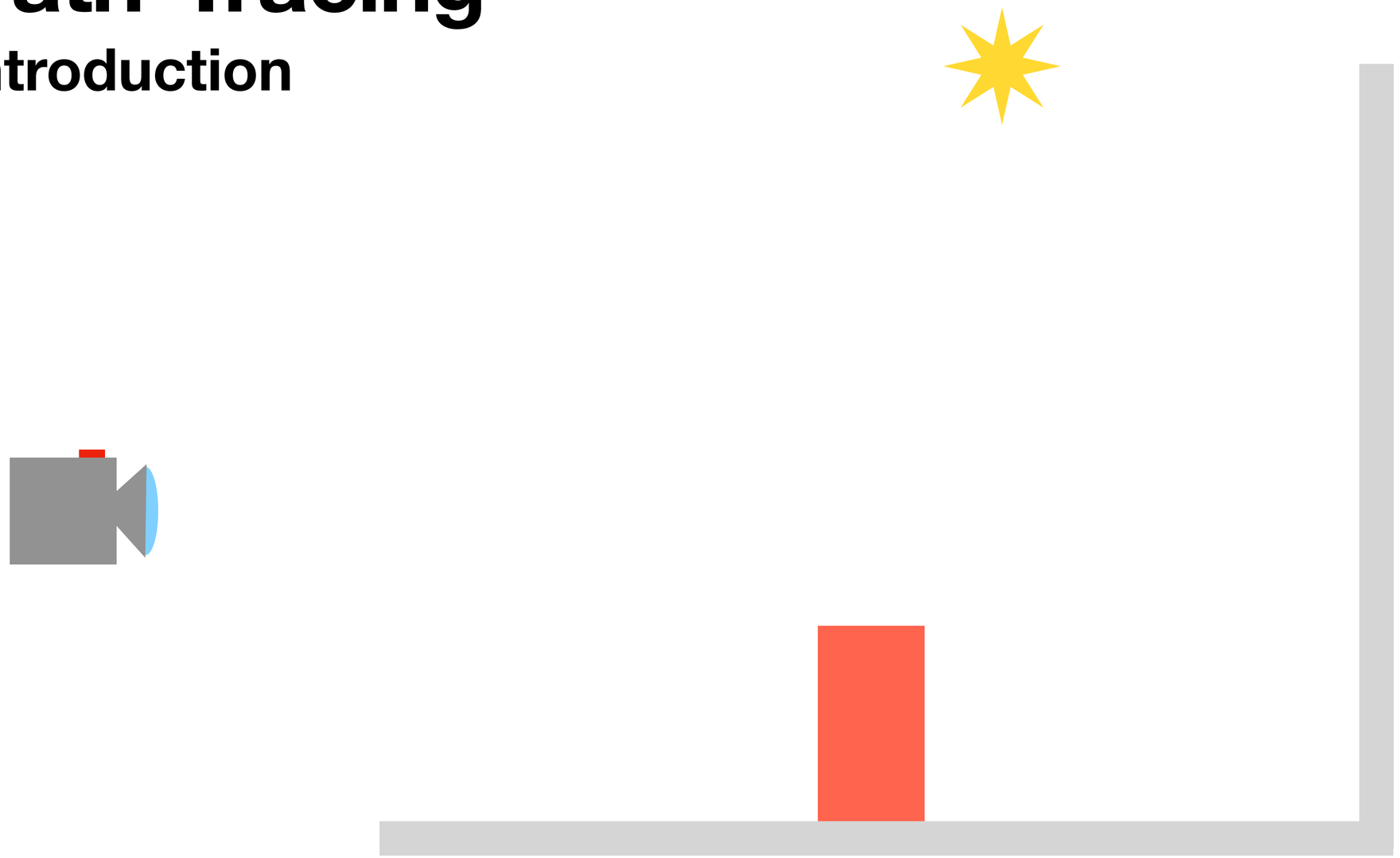
- Our estimator is the classic estimator seen so far:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{n} Y_i \, .$$

So We Generate Different Paths and We Sum Them Up

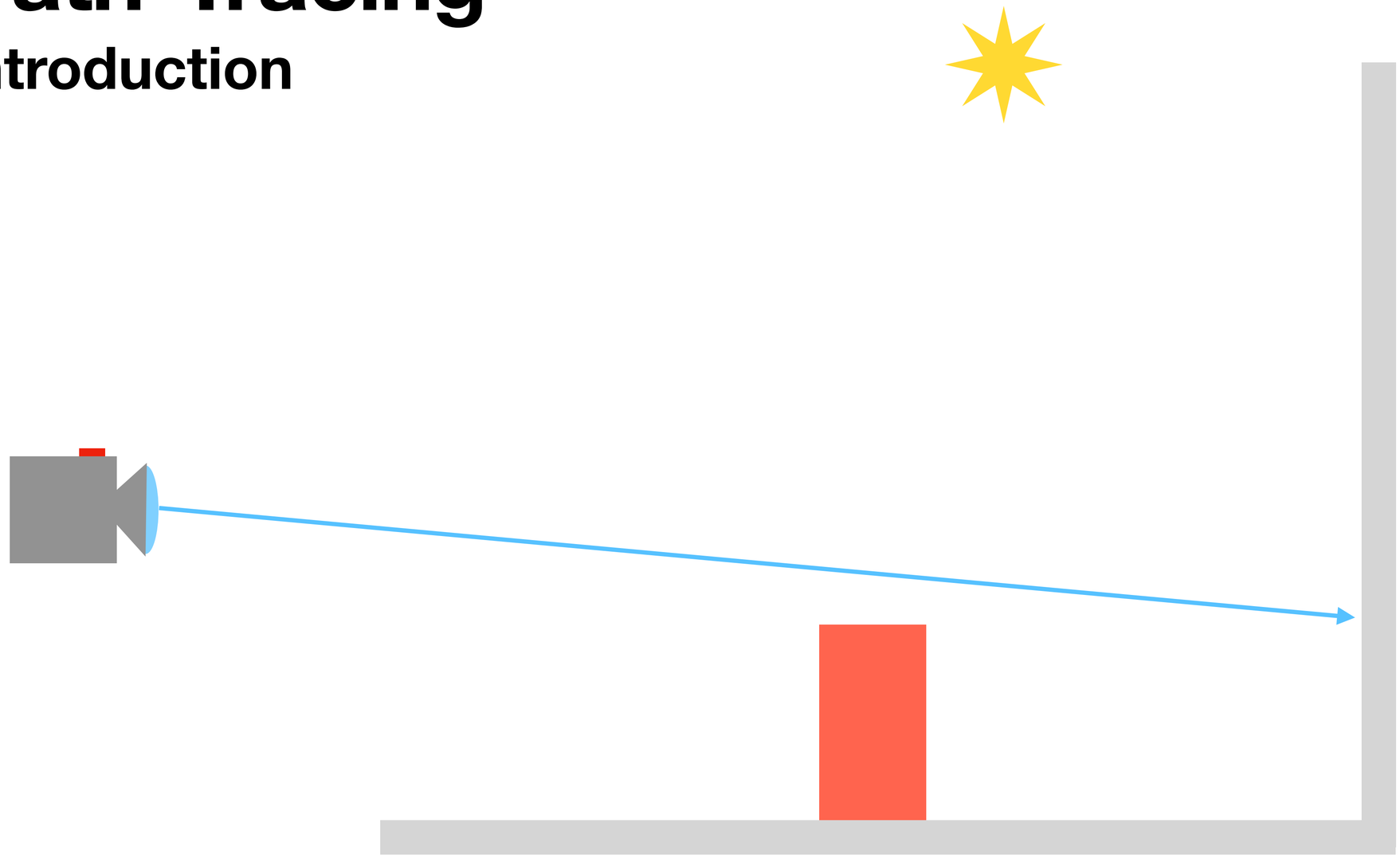
Path-Tracing

Introduction



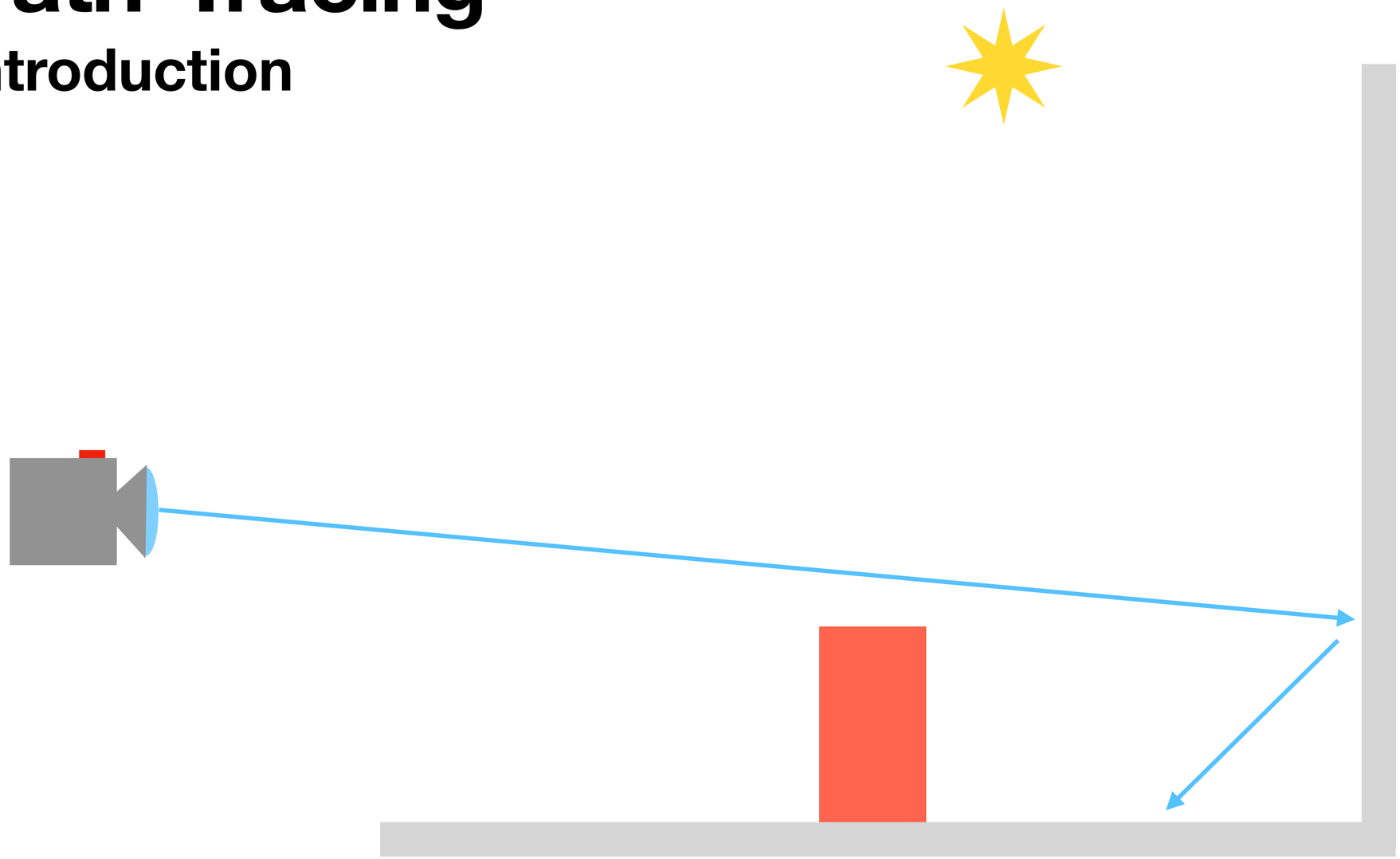
Path-Tracing

Introduction



Path-Tracing

Introduction



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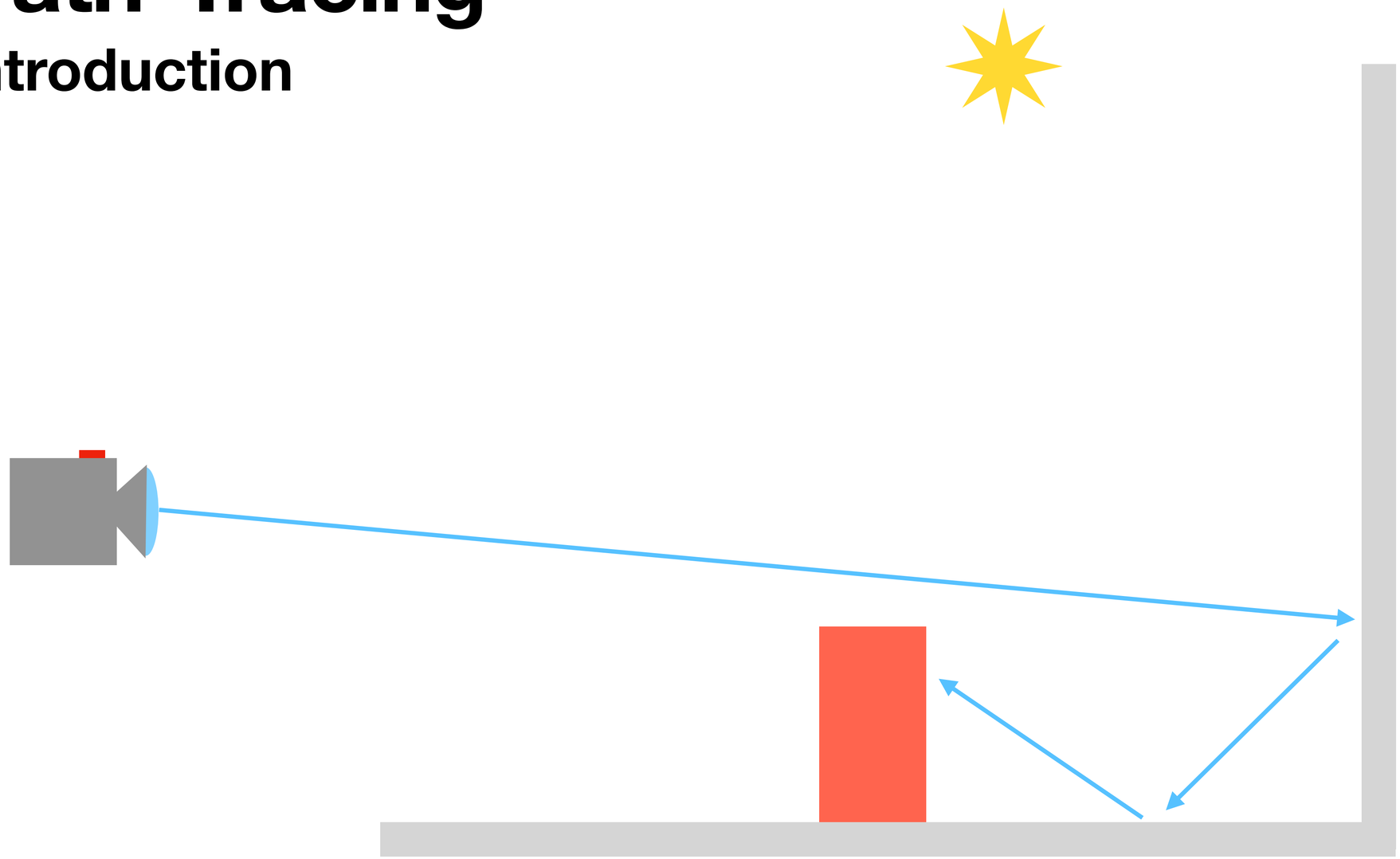

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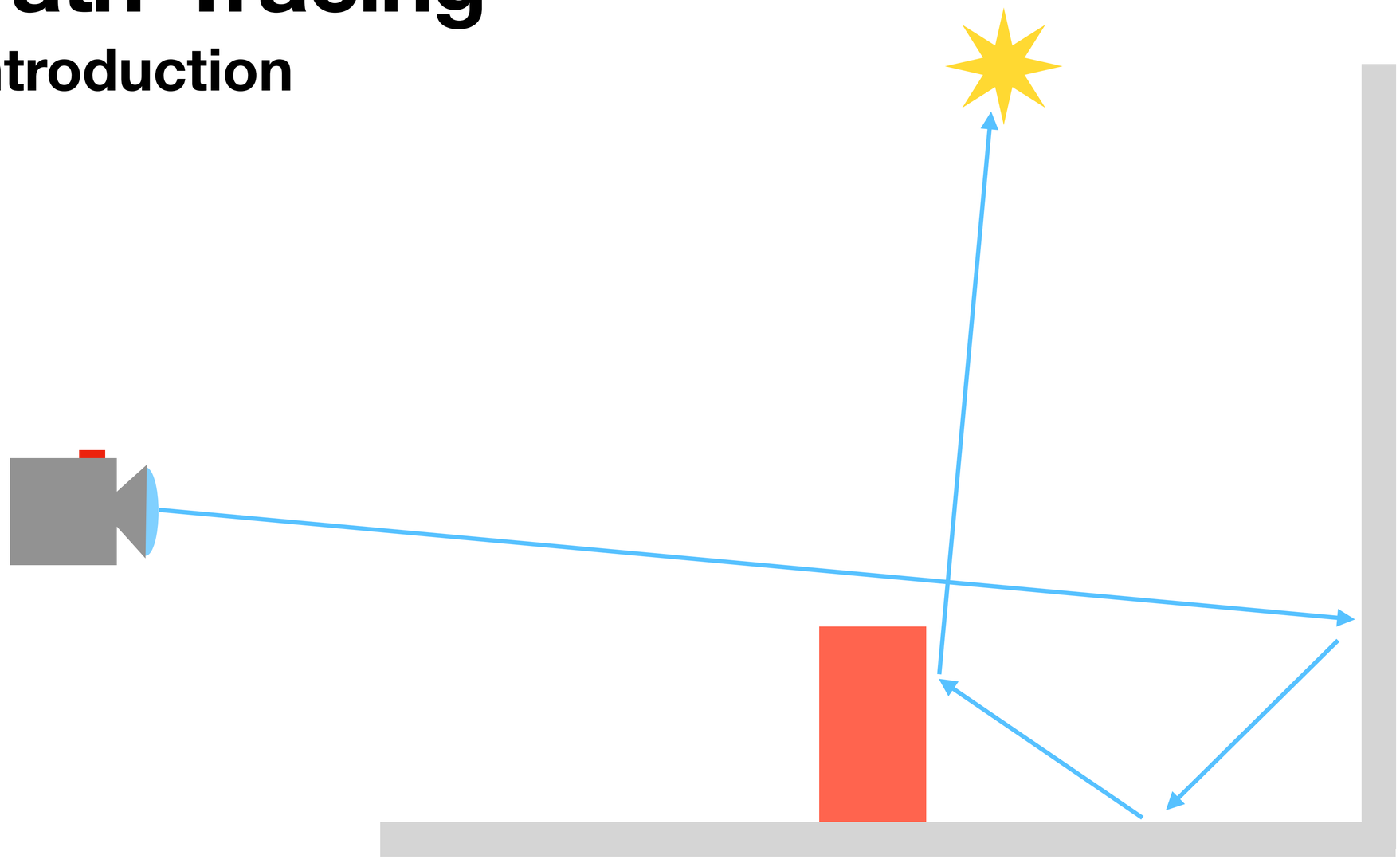

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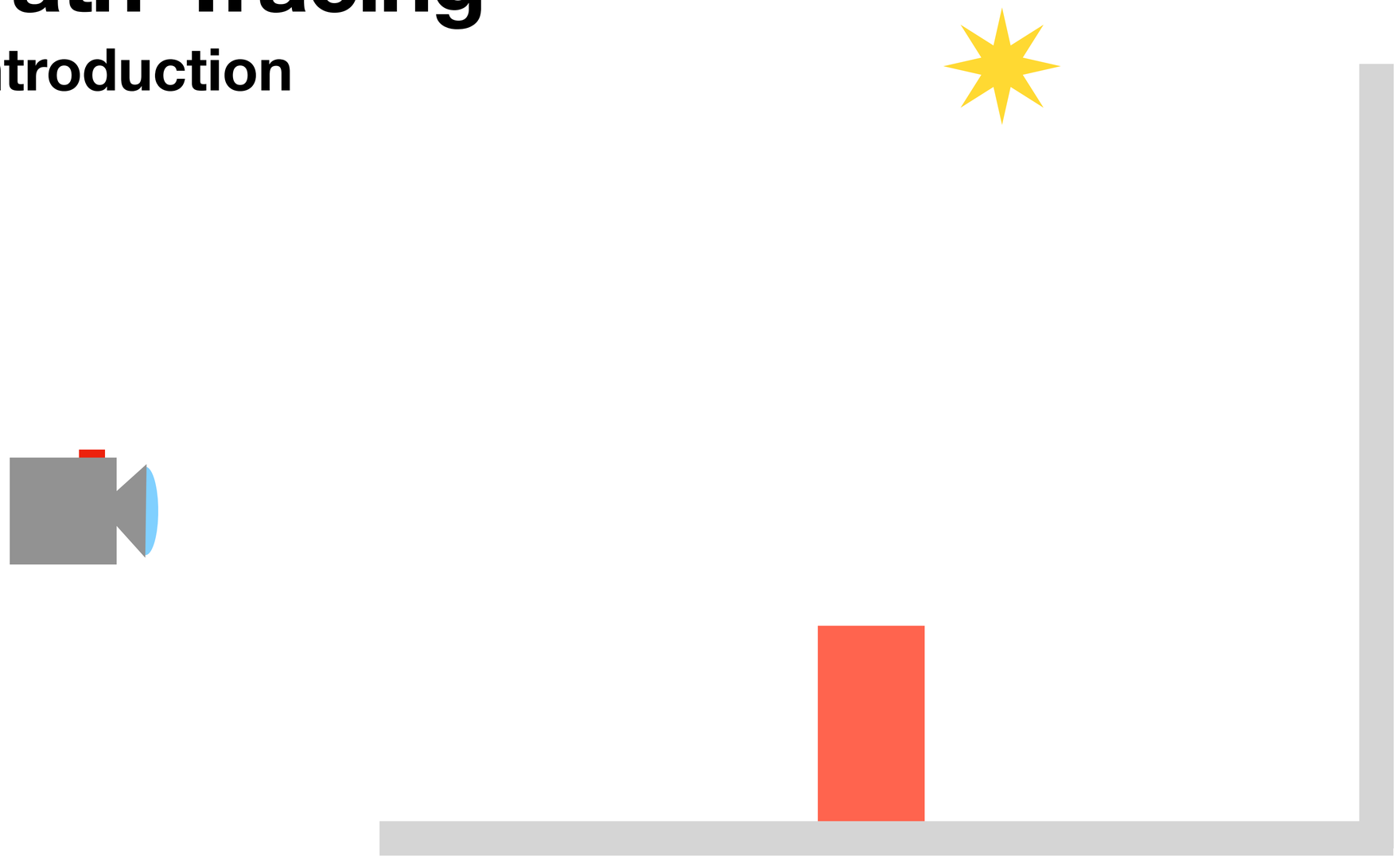

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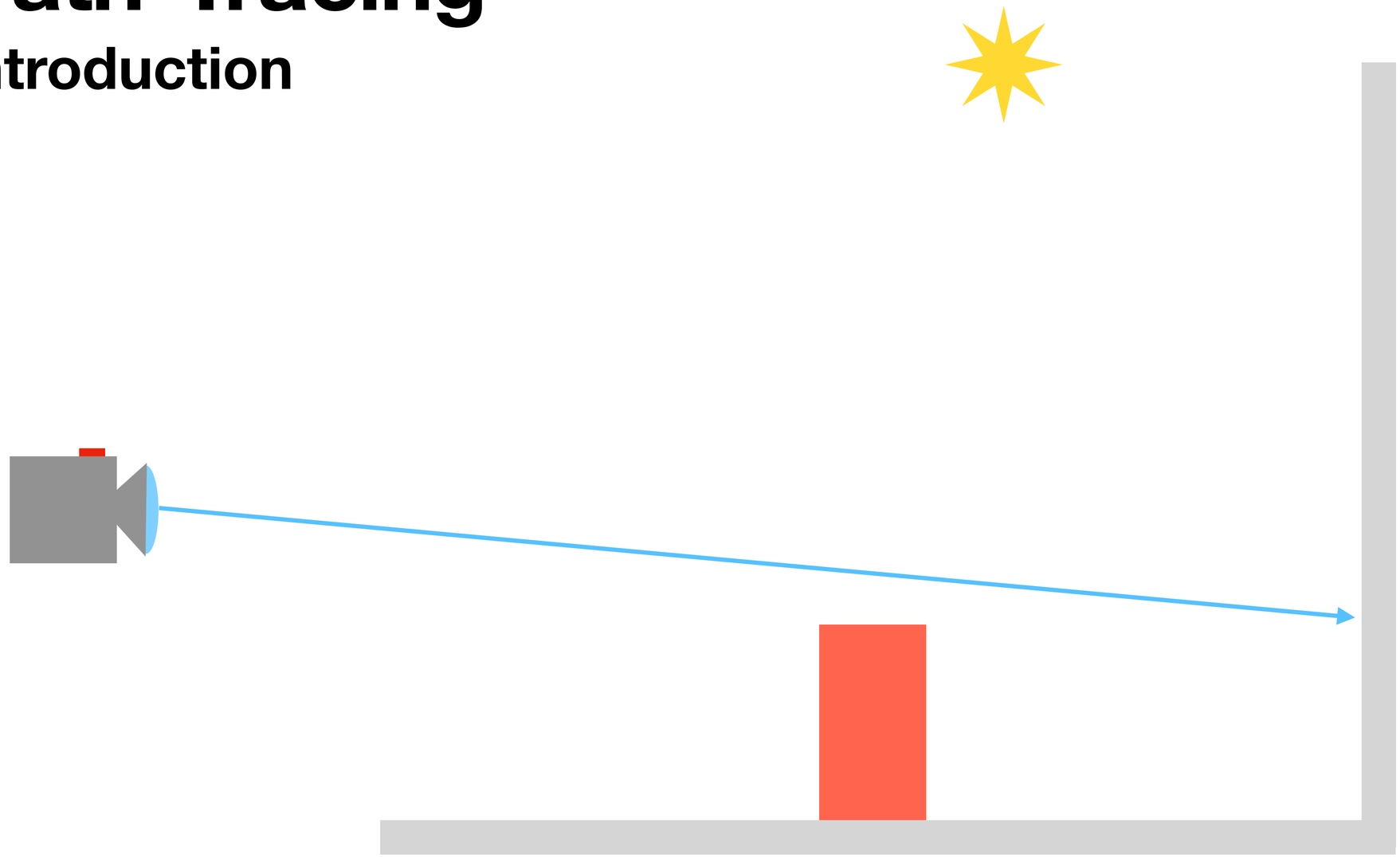

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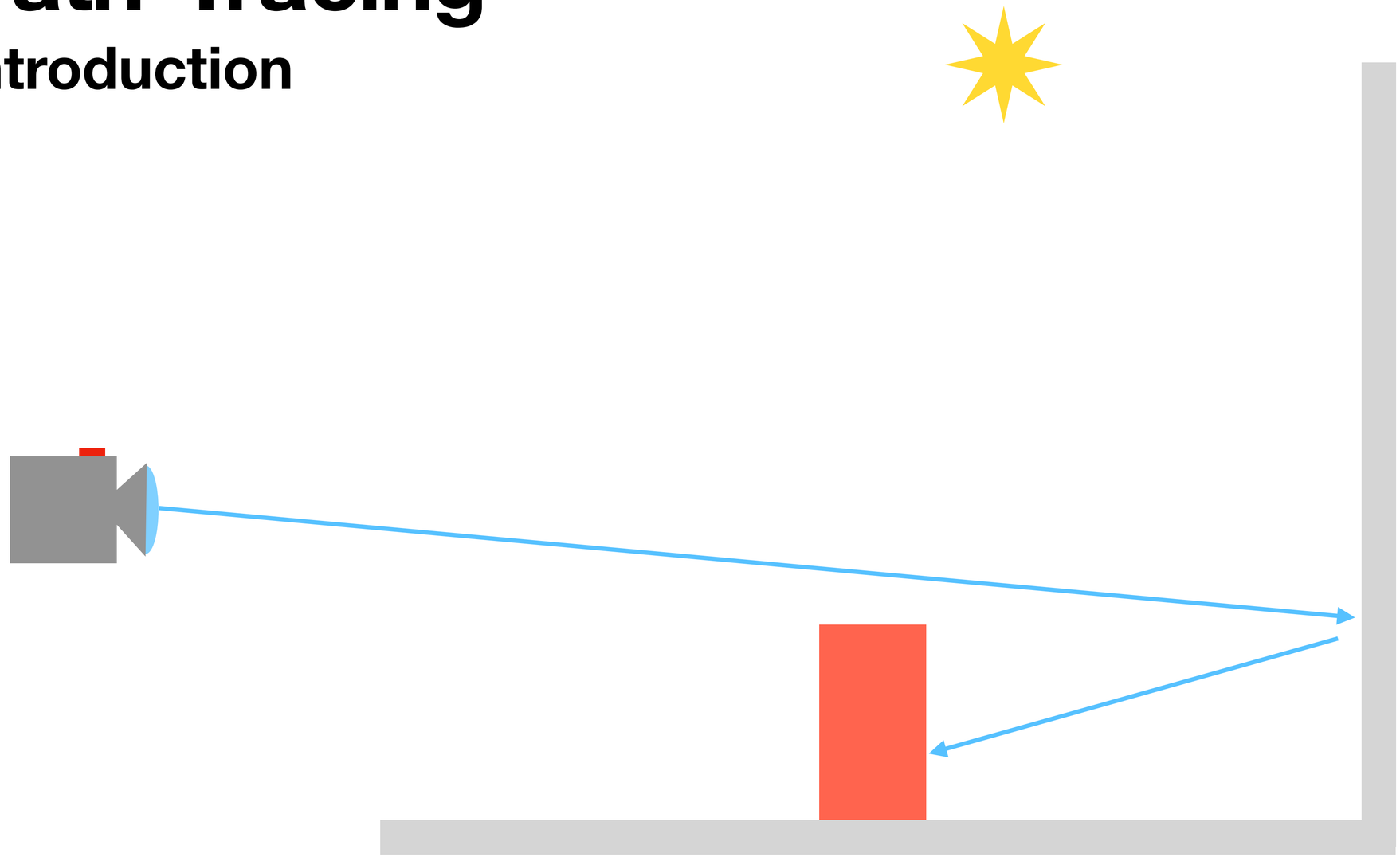

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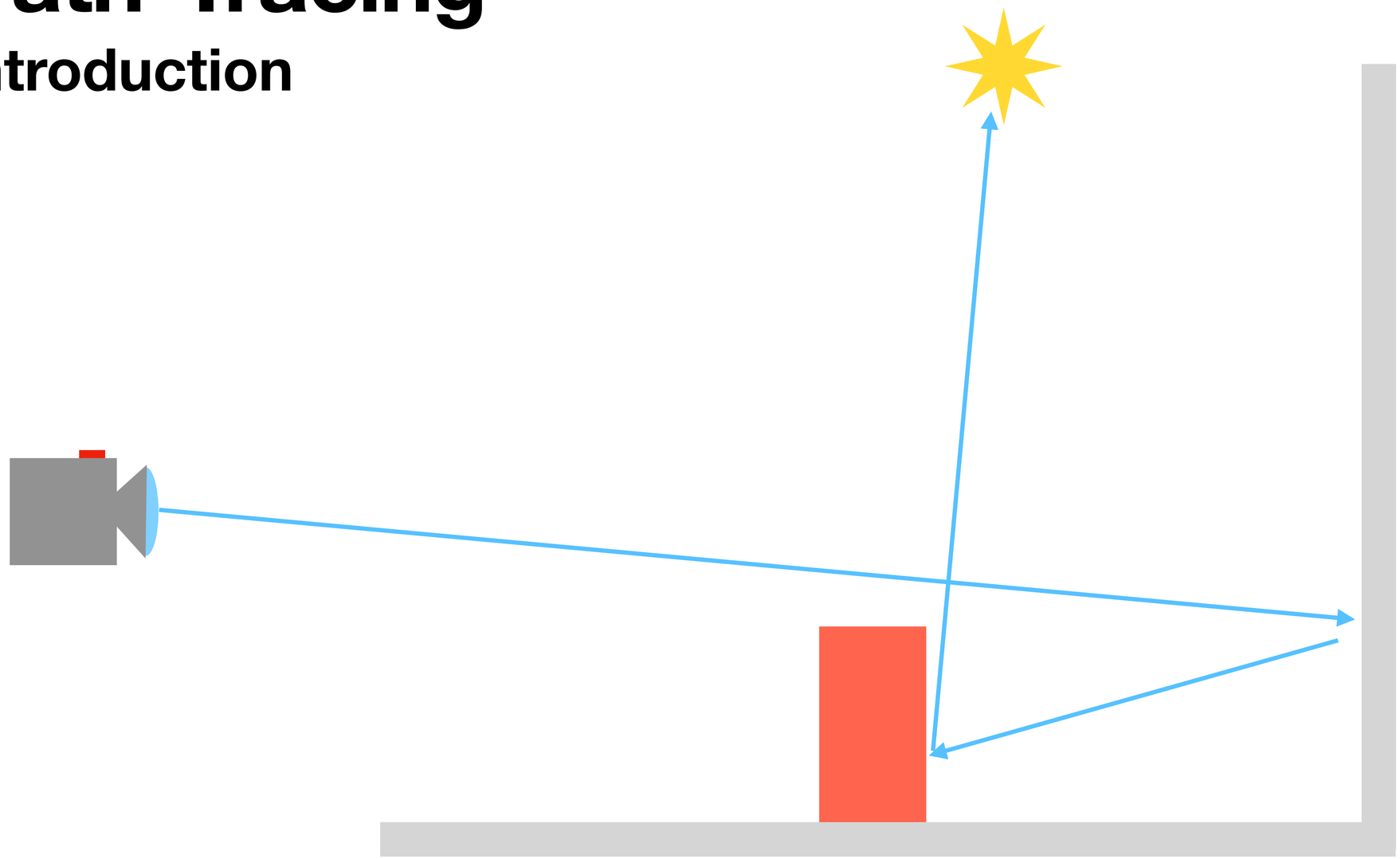

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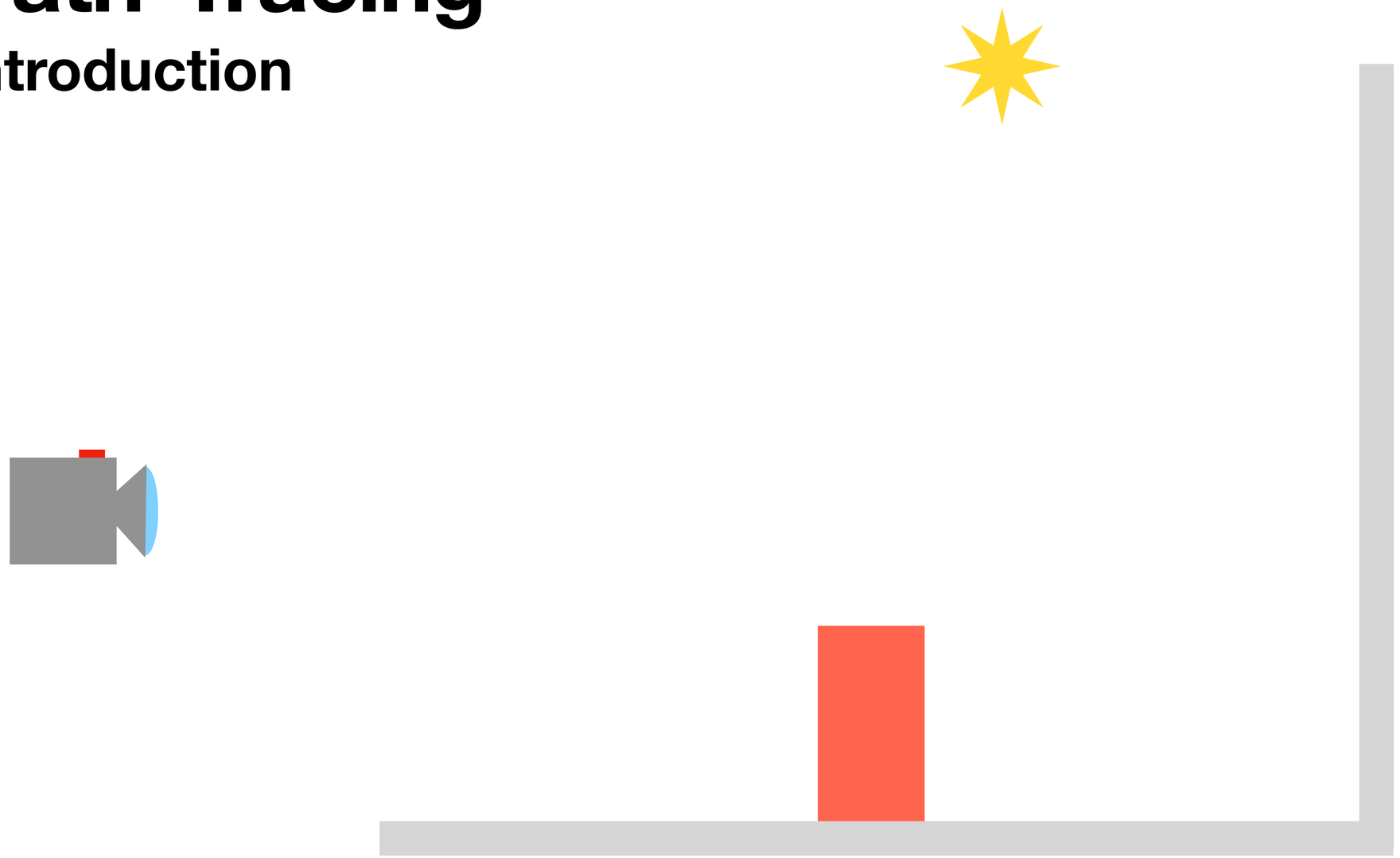

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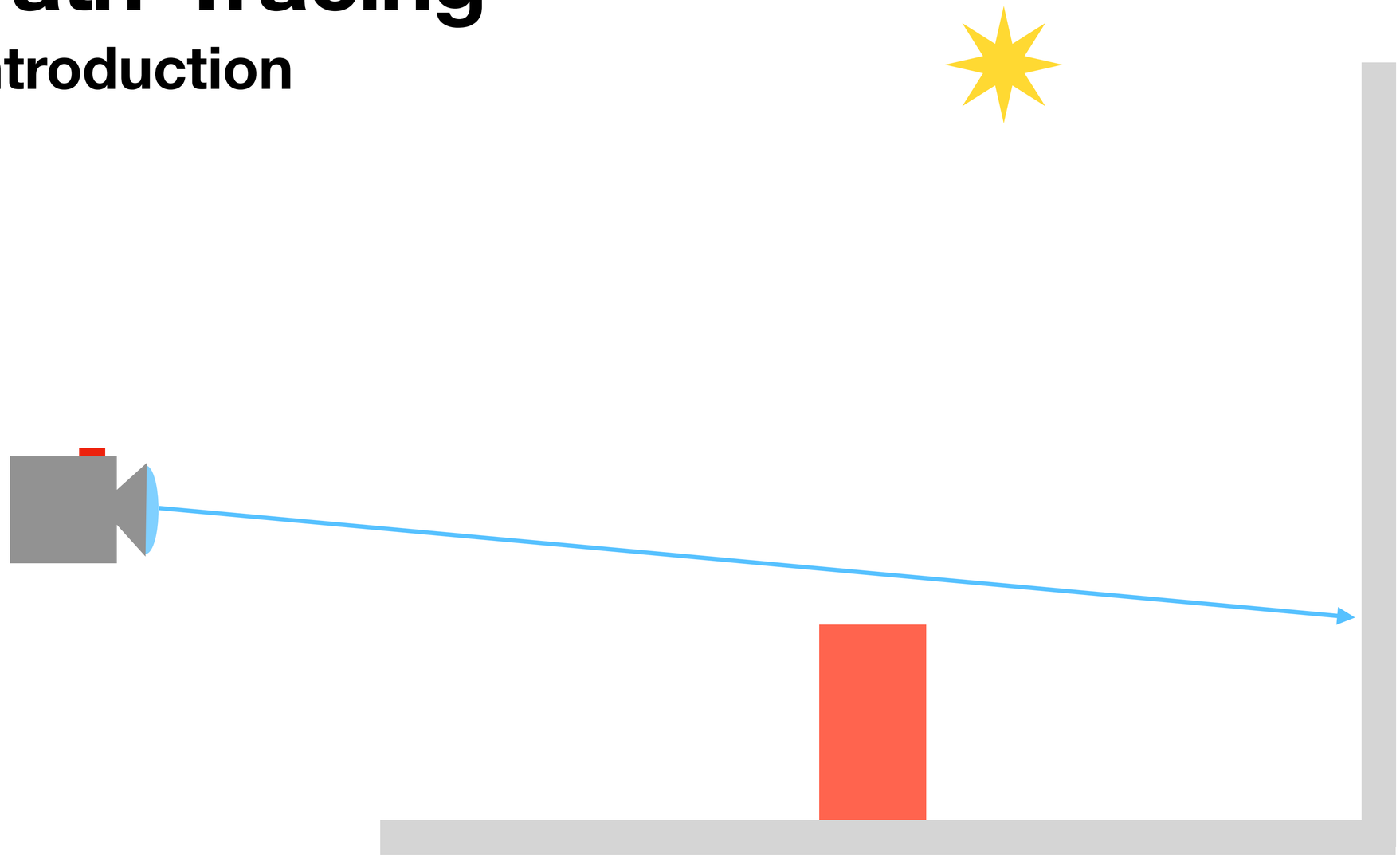

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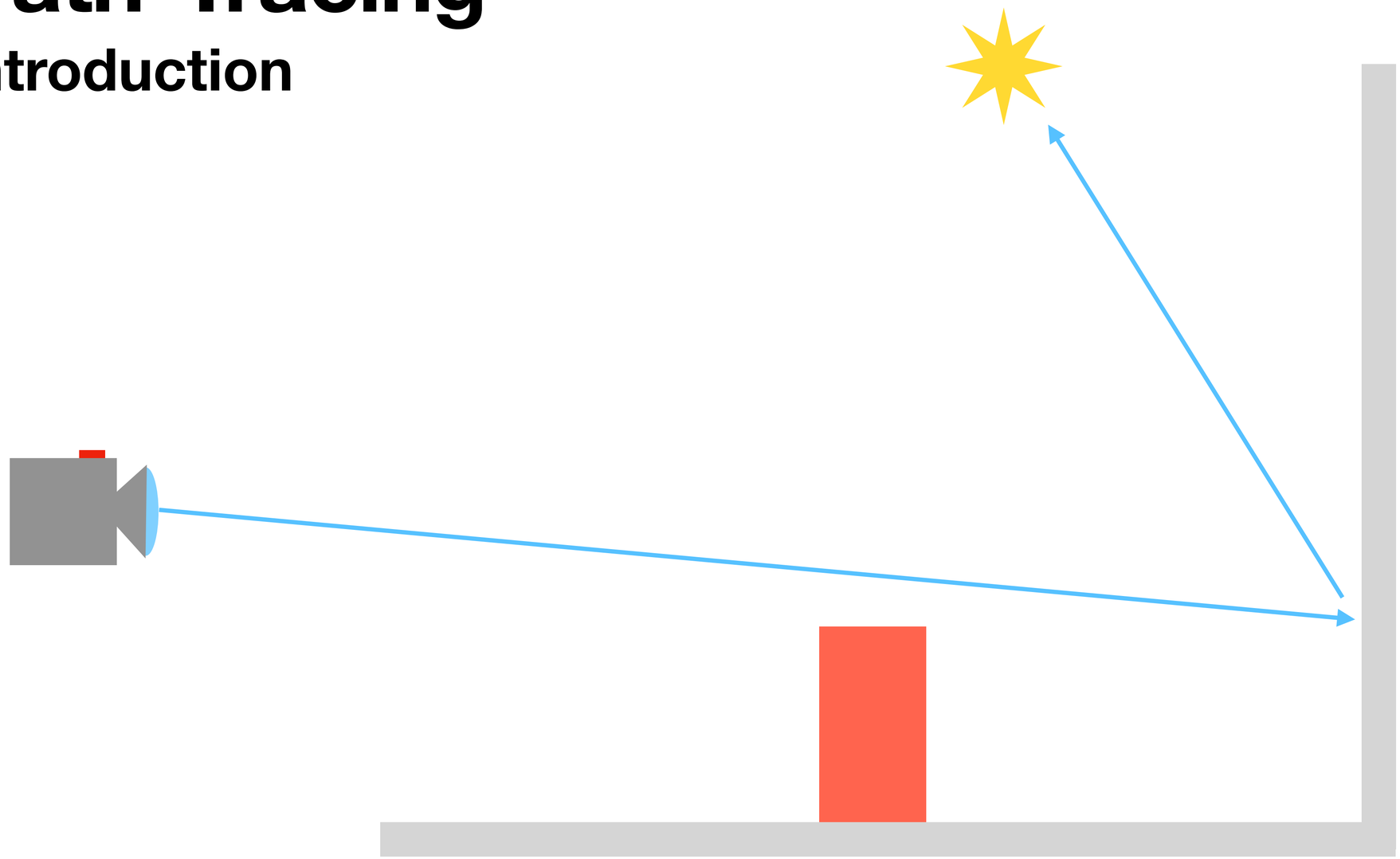

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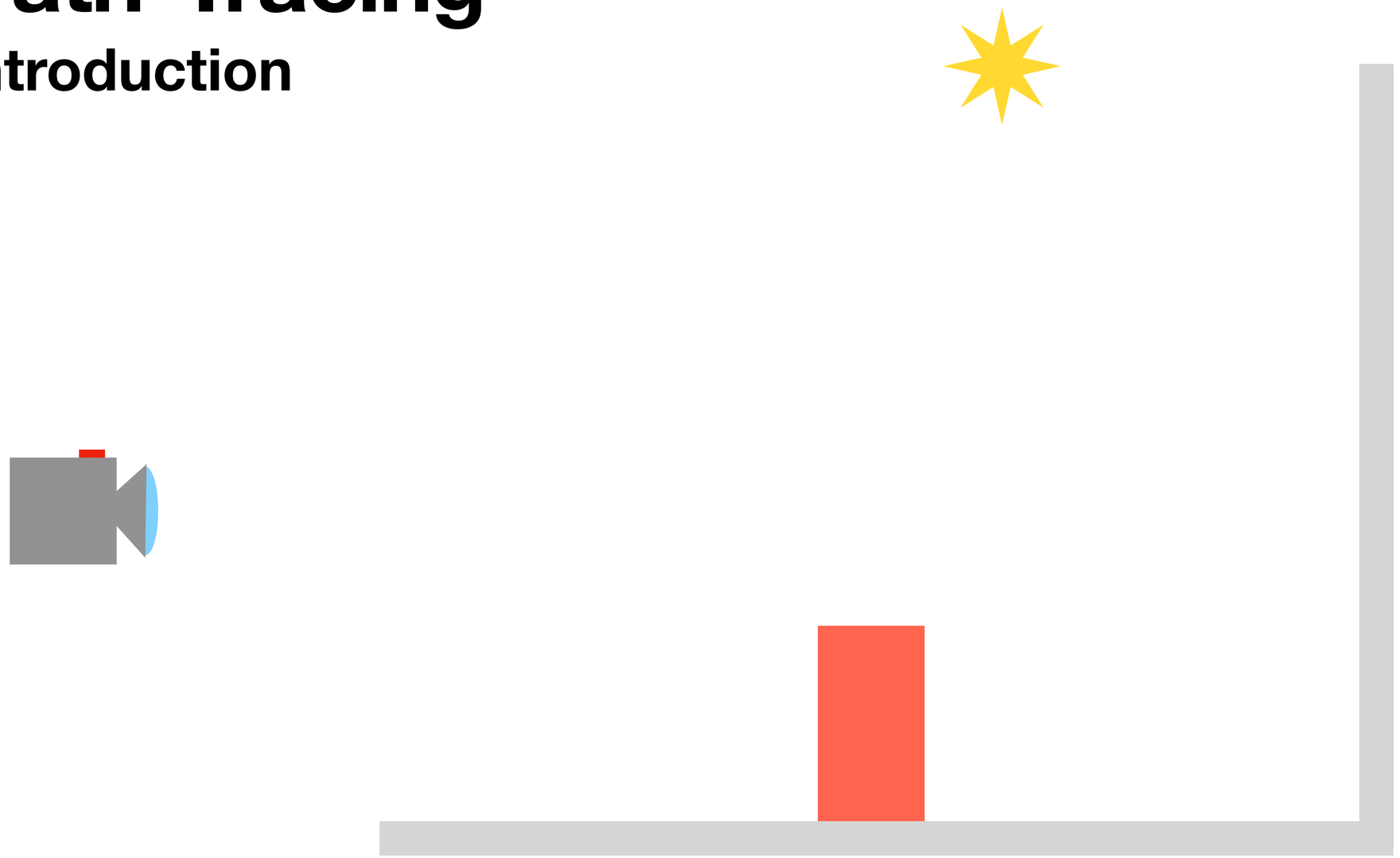

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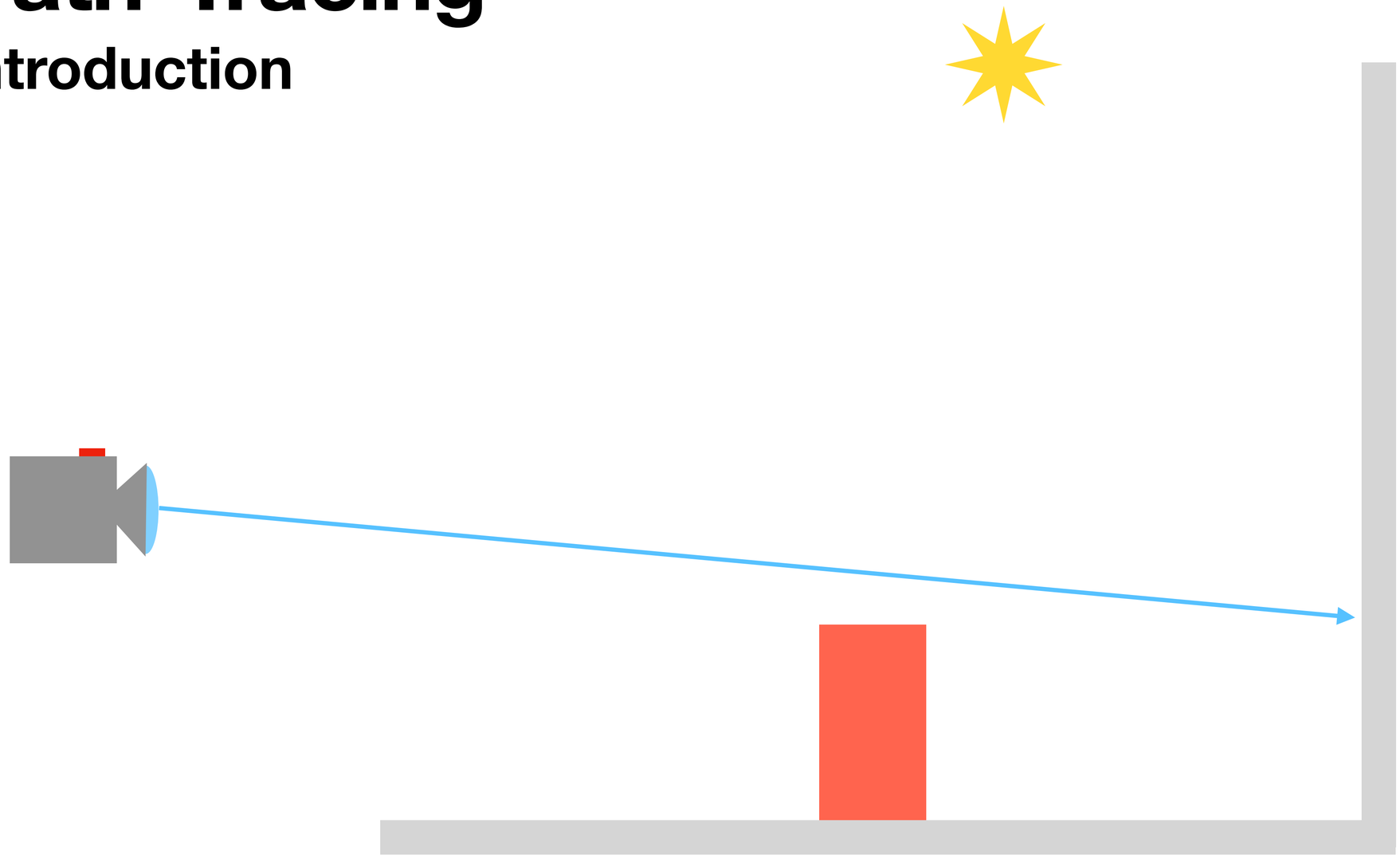

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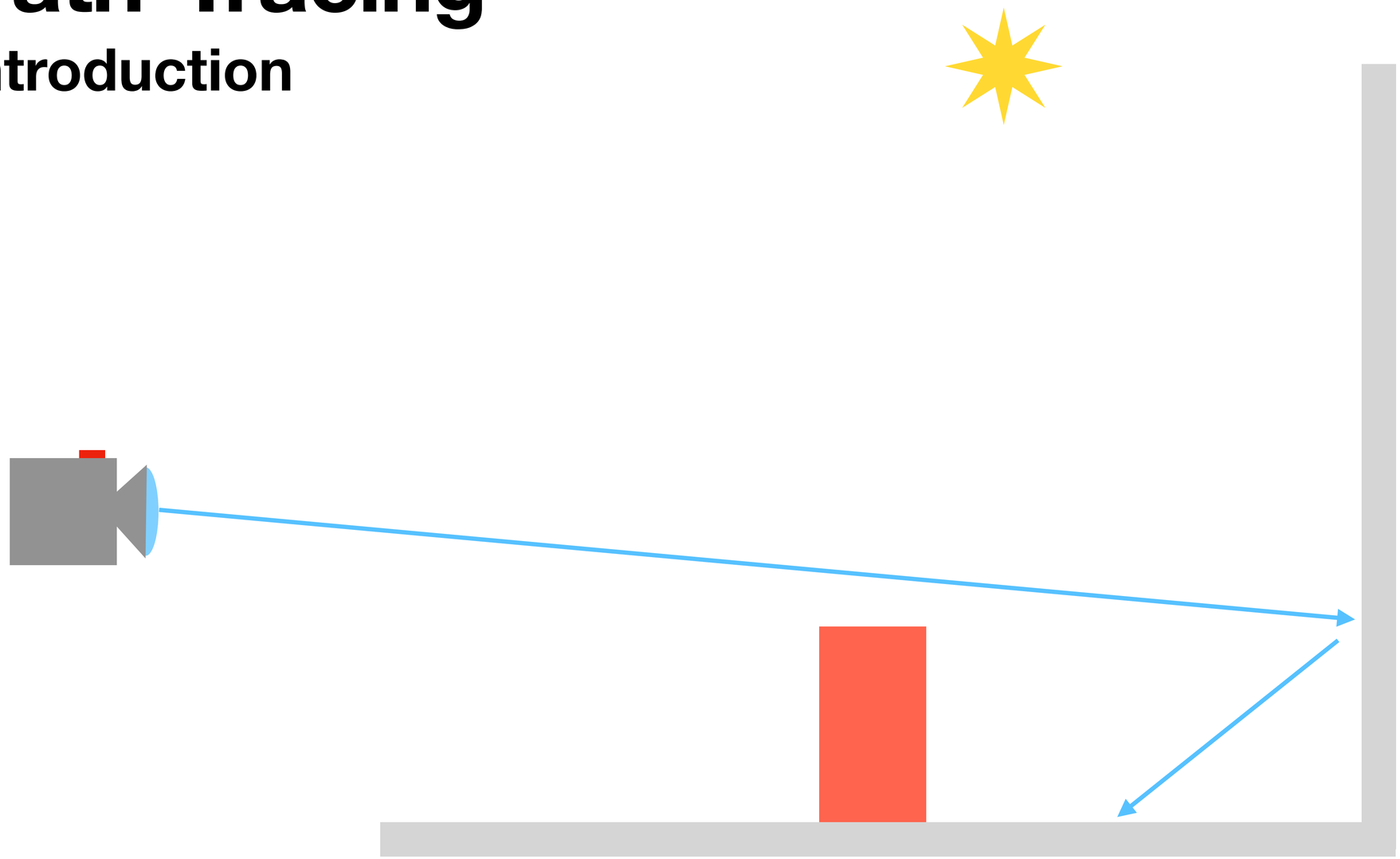

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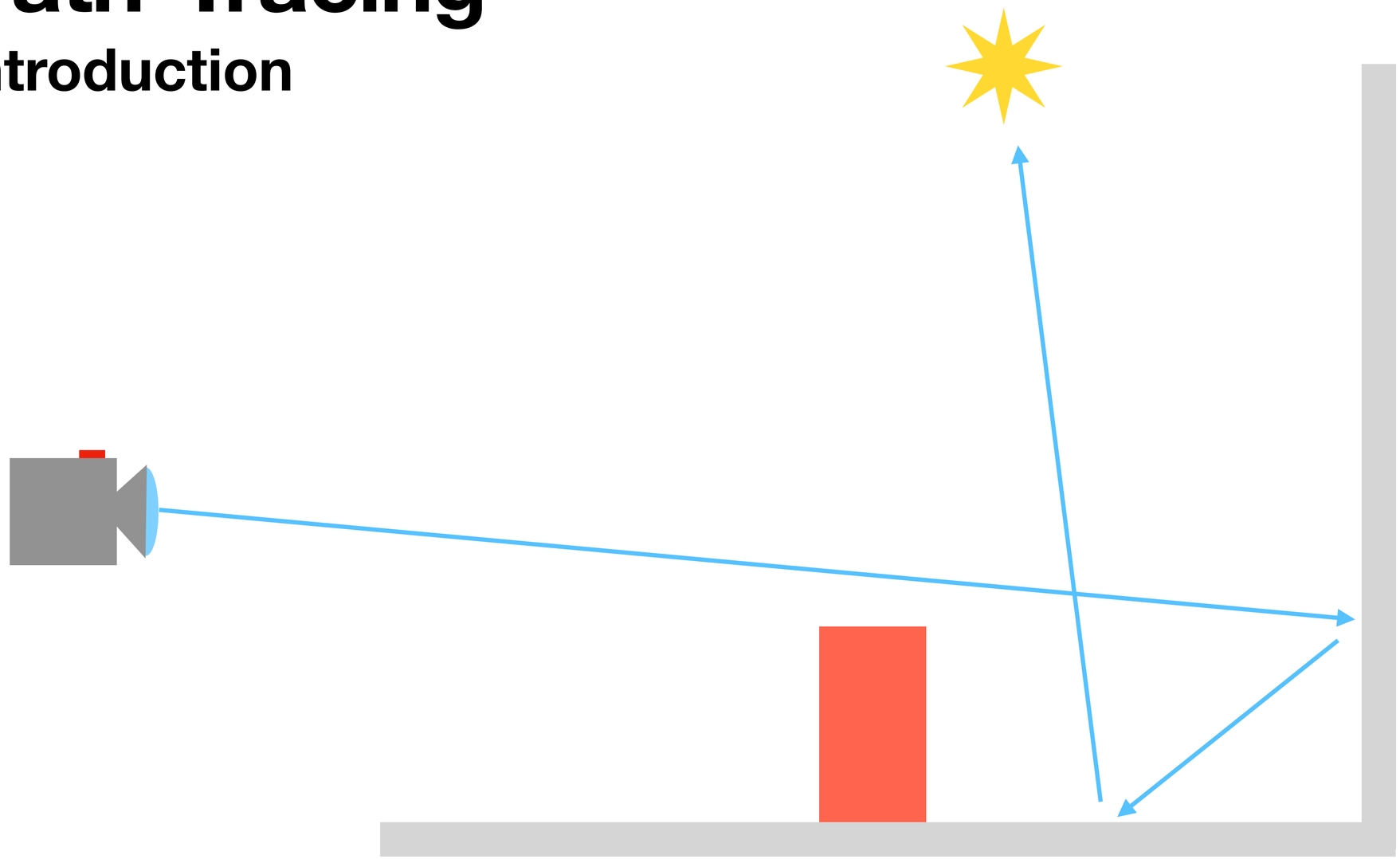


Path-Tracing

Monte-Carlo Techniques

- Techniques used:
  - Russian roulette —> to limit the length of paths.
  - Stratification.
  - Importance sampling:
    - 1D/2D distribution of light sources;
    - BRDF.
  - Metropolis.

Path-Tracing

Sampling the BRDF

- To sample the BRDF, we generate $\overrightarrow{\omega}_i$ directions randomly chosen according to its PDF:

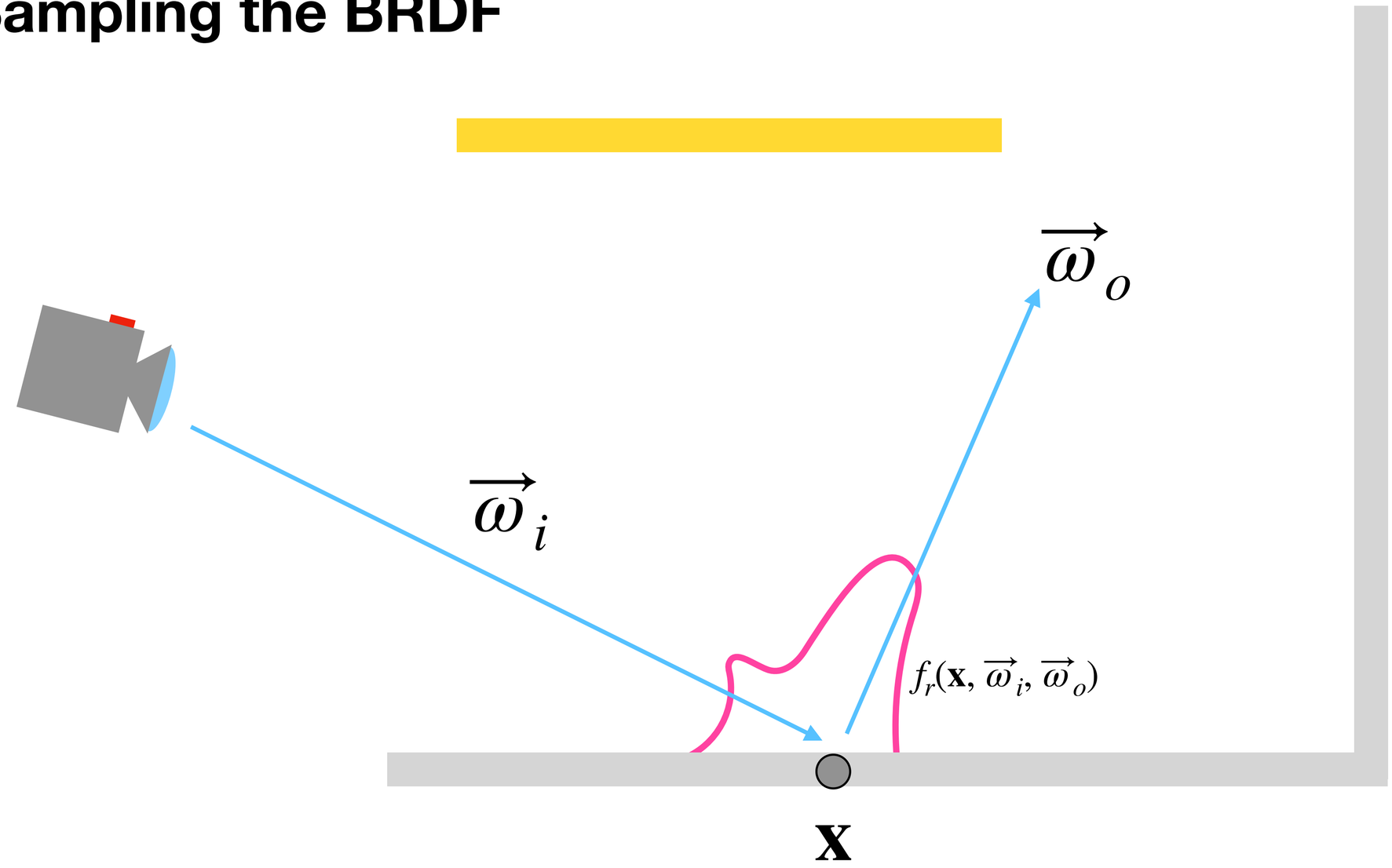
$$p(\overrightarrow{\omega}_i) \propto f_r(\mathbf{x}, \overrightarrow{\omega}_i, \overrightarrow{\omega}_o).$$

- So, we compute our estimate as:

$$L_o(\mathbf{x}, \overrightarrow{\omega}_o) \approx \frac{f_r(\mathbf{x}, \overrightarrow{\omega}_i, \overrightarrow{\omega}_o) L_i(\mathbf{x}, \overrightarrow{\omega}_i)}{p(\overrightarrow{\omega}_i)}.$$

Path-Tracing

Sampling the BRDF



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Sampling the BRDF

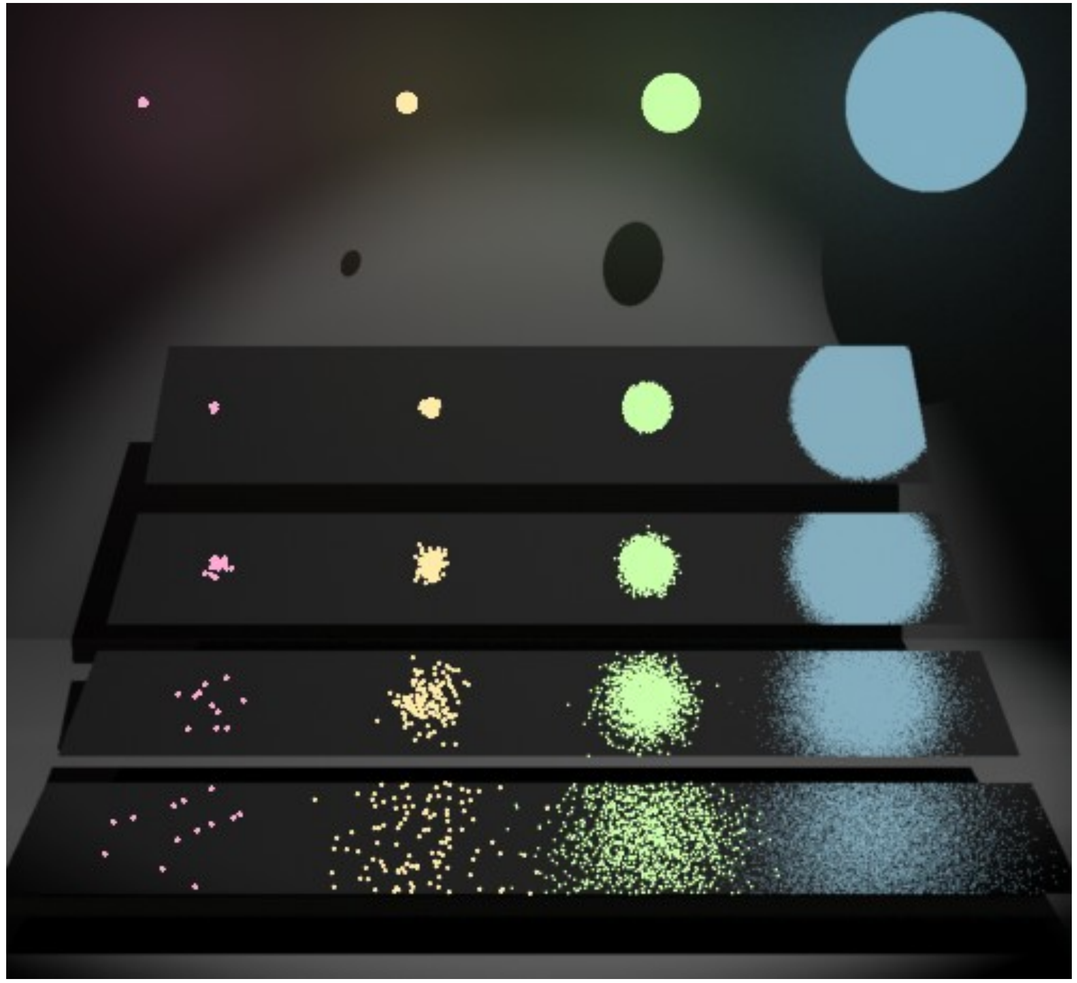


Image by Eric Veach

]
Sampling the Light Source

- To sample the light source, we generate random points, $\mathbf{x}_l$, on the light source according to its PDF:

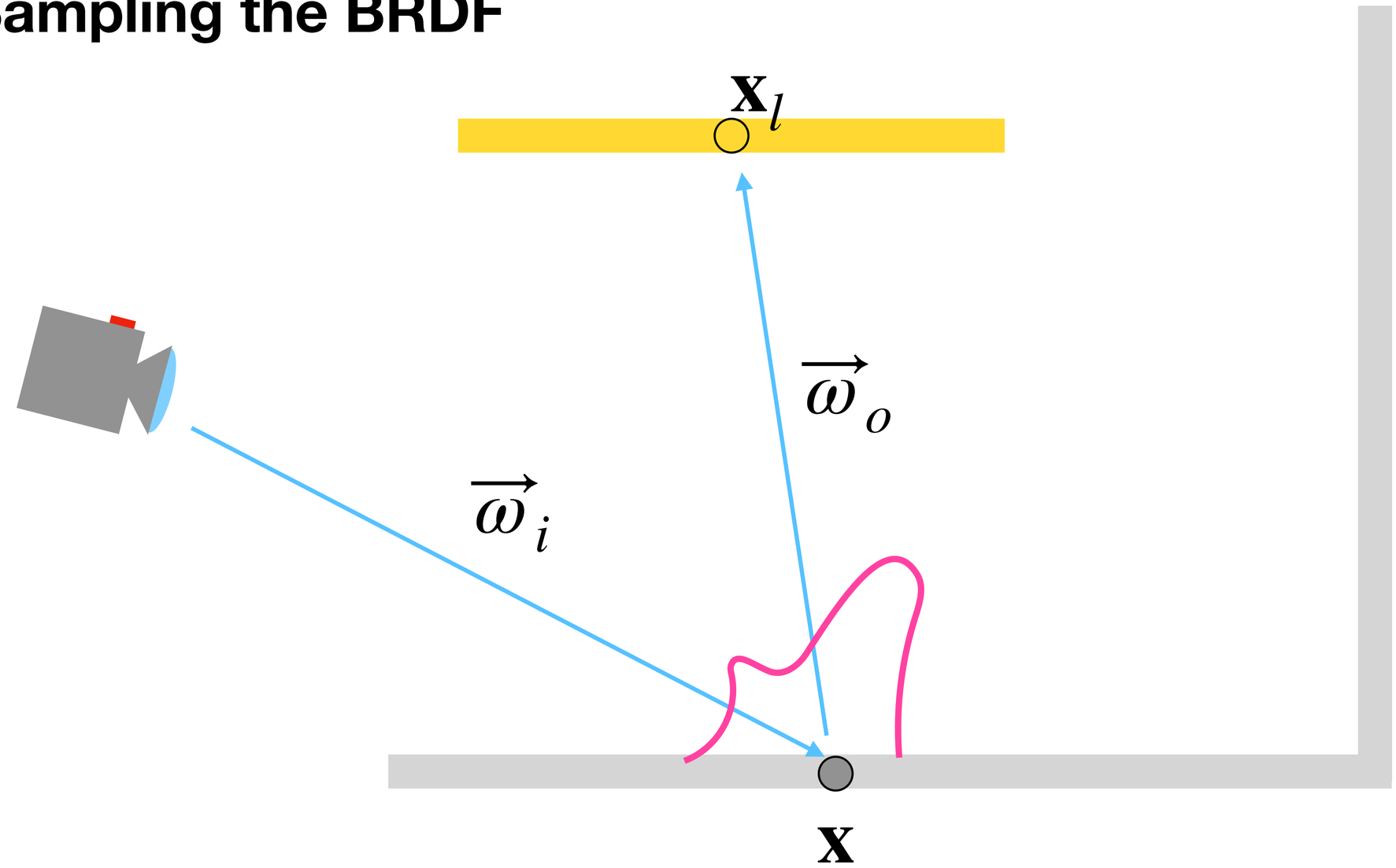
$$p(\mathbf{x}_l).$$

- So, we compute our estimate as:

$$L_o(\mathbf{x}, \overrightarrow{\omega}_o) \approx \frac{f_r(\mathbf{x}, \overrightarrow{\omega}'_i, \overrightarrow{\omega}_o) L_i(\mathbf{x}, \overrightarrow{\omega}'_i)}{p(\mathbf{x}_l)} \qquad \overrightarrow{\omega}'_i = \frac{\mathbf{x}_l - \mathbf{x}}{\|\mathbf{x}_l - \mathbf{x}\|}.$$

Path-Tracing

Sampling the BRDF



Path-Tracing

Sampling the Light Source

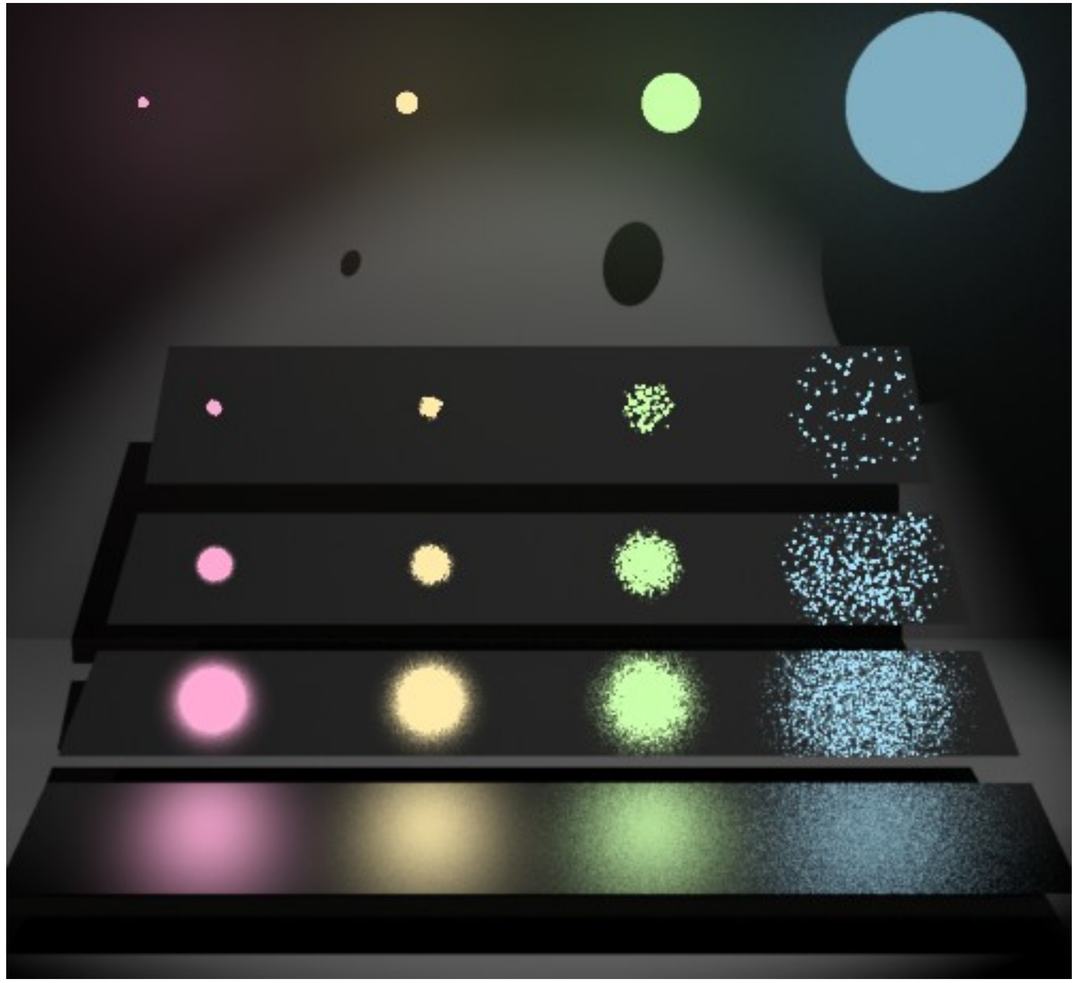


Image by Eric Veach

Path-Tracing

Multiple Importance Sampling (MIS)

- The naive solution would be to average the two estimations:
  - However, variance is additive, so we do not decrease it!
- The main idea of Multiple Importance Sampling (MIS) is to:
  - Draw samples from different distributions;
  - Mix all these samples using weights:
    - These weights should remove large peaks of variance when we have differences between our estimation and the distribution.

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Path-Tracing
MIS

- In general, we may have K distributions, q_i, and we generate n_j samples $\mathbf{x}_{i,j} \sim q_j$ for each distribution.
- In this case, our estimation is:

$$\hat{\mu} = \sum_{j=1}^{K} \frac{1}{n_j} \sum_{i=1}^{n_j} \omega_j(\mathbf{x}_{i,j}) \frac{f(\mathbf{x}_{i,j}) p(\mathbf{x}_{i,j})}{q_j(\mathbf{x}_{i,j})}.$$

- The weighting function, $\omega(\mathbf{x}) \geq 0$, is normalized:

$$\sum_{j=1}^{K} \omega(\mathbf{x}) = 1.$$

- Balance heuristic $\omega_j(\mathbf{x}) \propto n_j q_j(\mathbf{x})$:

$$\omega_j(\mathbf{x}) = \frac{n_j q_j(\mathbf{x})}{\sum_{i=1}^{K} n_i q_i(\mathbf{x})}.$$

Path-Tracing
MIS

- What's about its variance?

$$\mathrm{Var}(\hat{\mu}_{BH}) = \mathrm{Var}(\hat{\mu}) + \left(\frac{1}{\min_j n_j} - \frac{1}{\sum_j n_j} \right) \mu^2.$$

- Other heuristics? Yes
  - Power Heuristic: $\omega_j(\mathbf{x}) \propto \left(n_j q_j(\mathbf{x}) \right)^\beta \quad \beta \geq 1.$

Path-Tracing

MIS Example

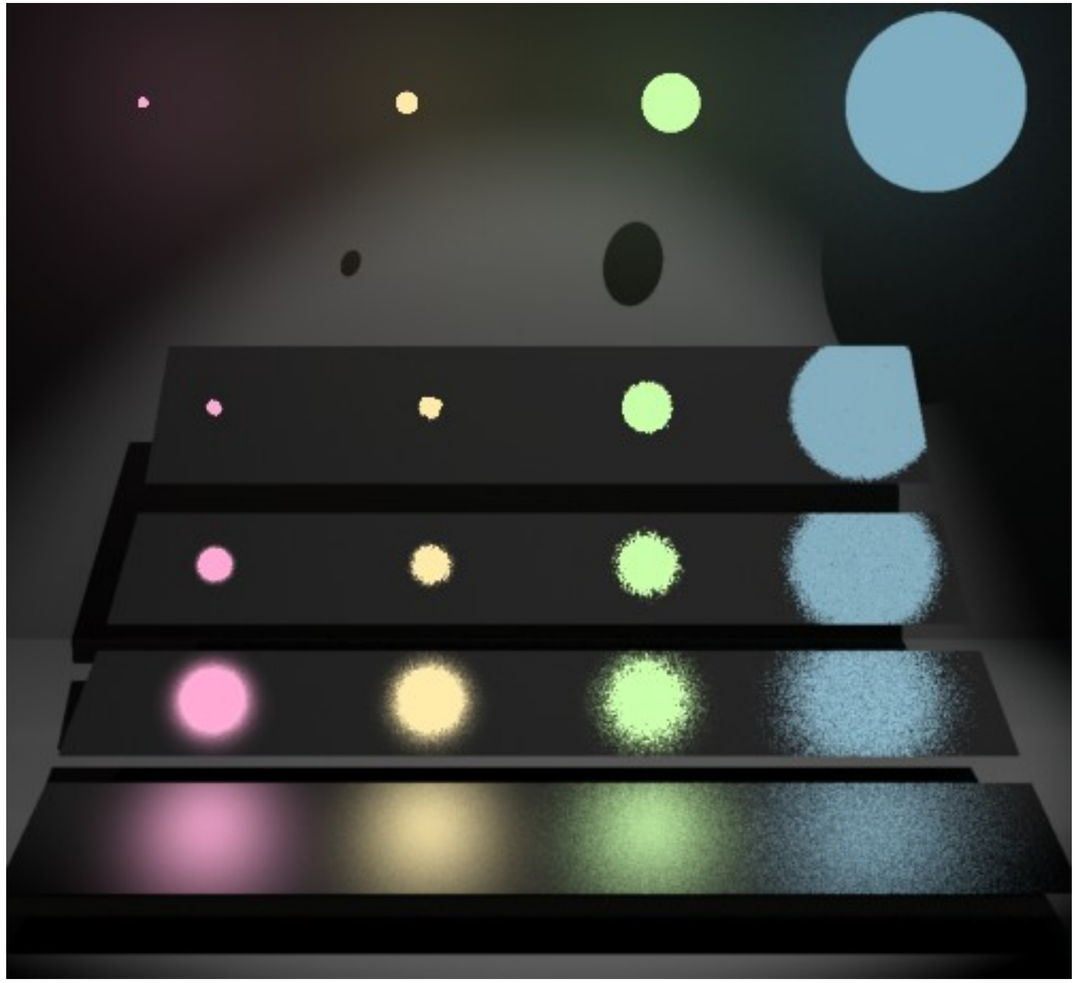


Image by Eric Veach

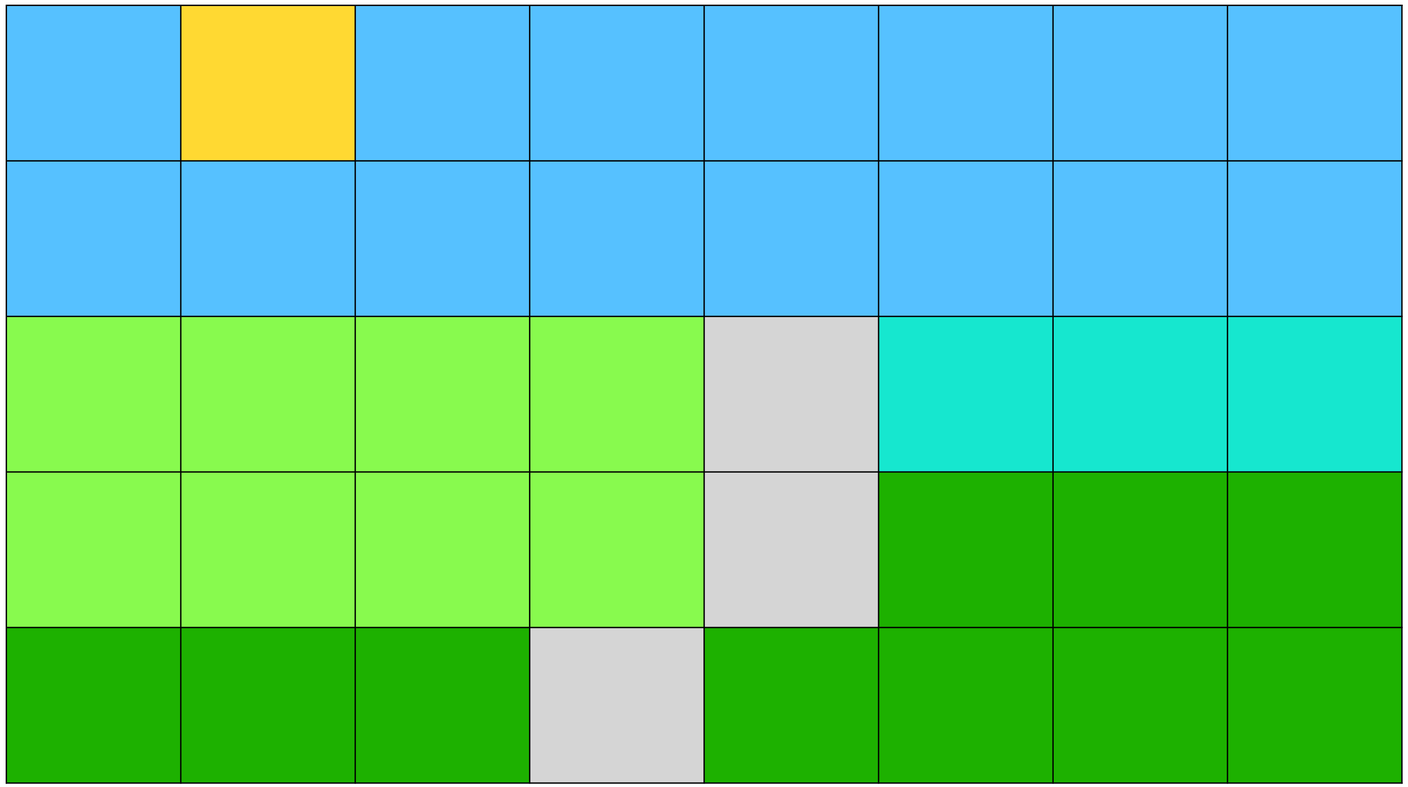
Path-Tracing

Complex Light Sources

- Typically, to have convincing light sources, they have to be spatially varying possibly using scene-referred photographs (i.e., calibrated):

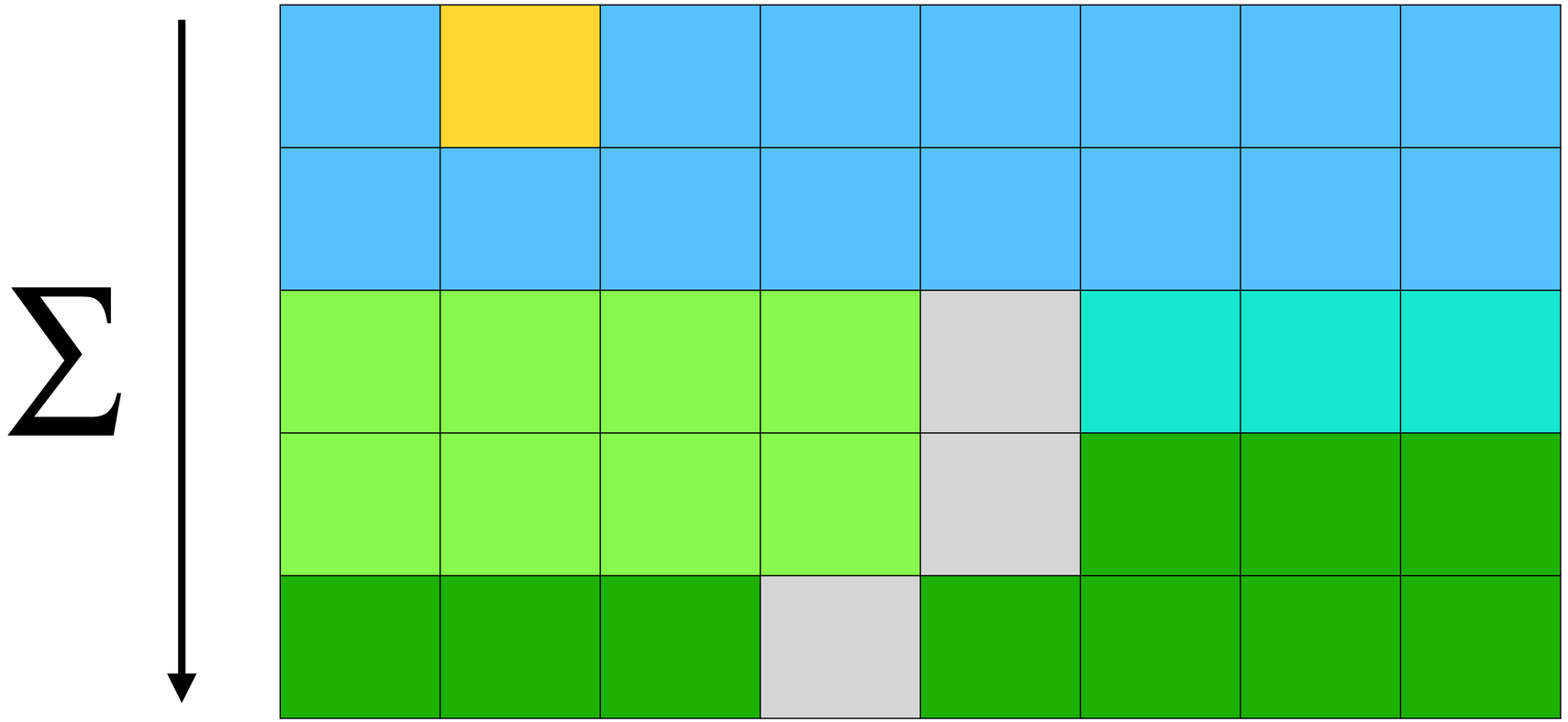
Path-Tracing

Complex Light Sources



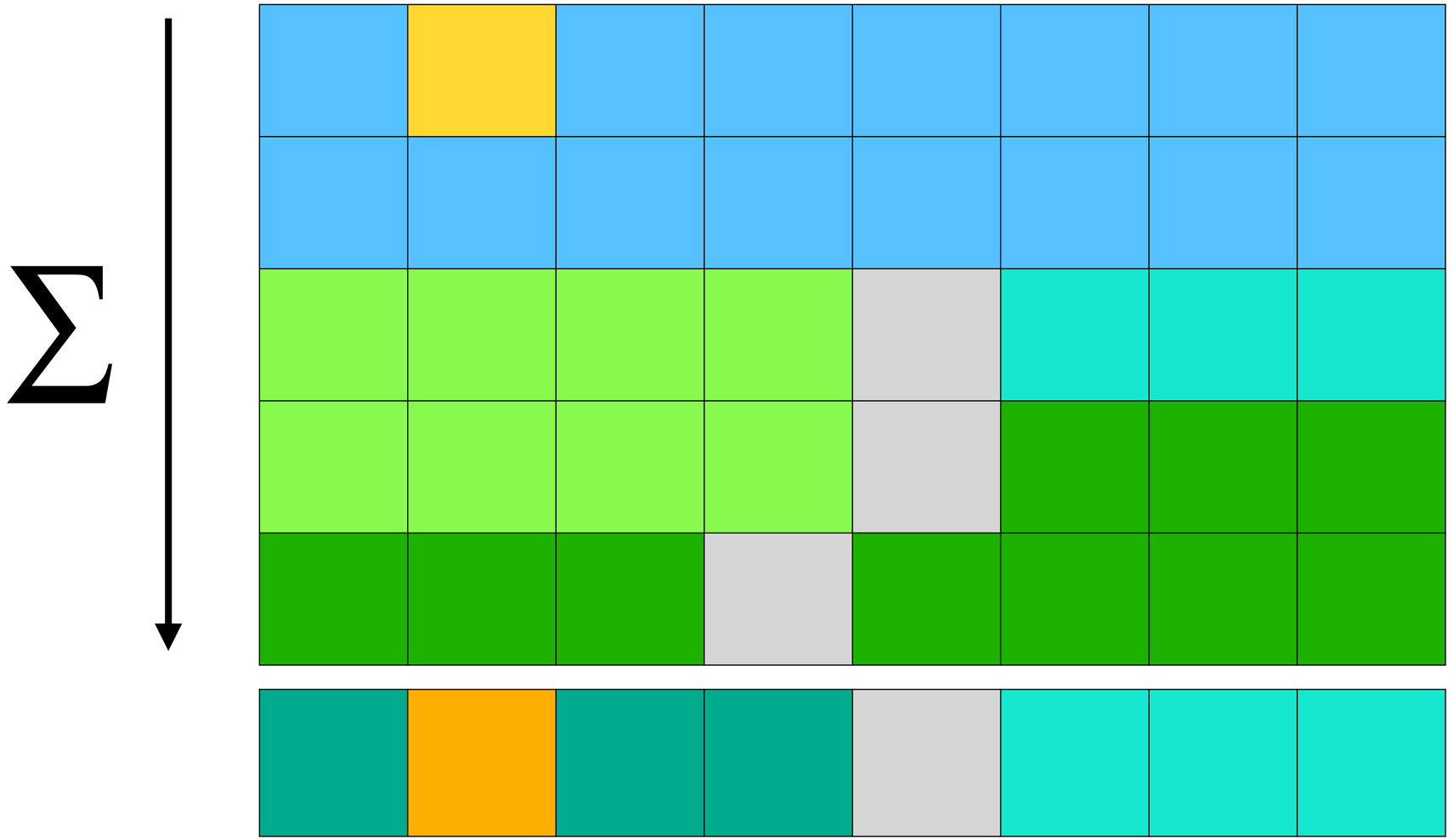
Path-Tracing

Complex Light Sources



Path-Tracing

Complex Light Sources



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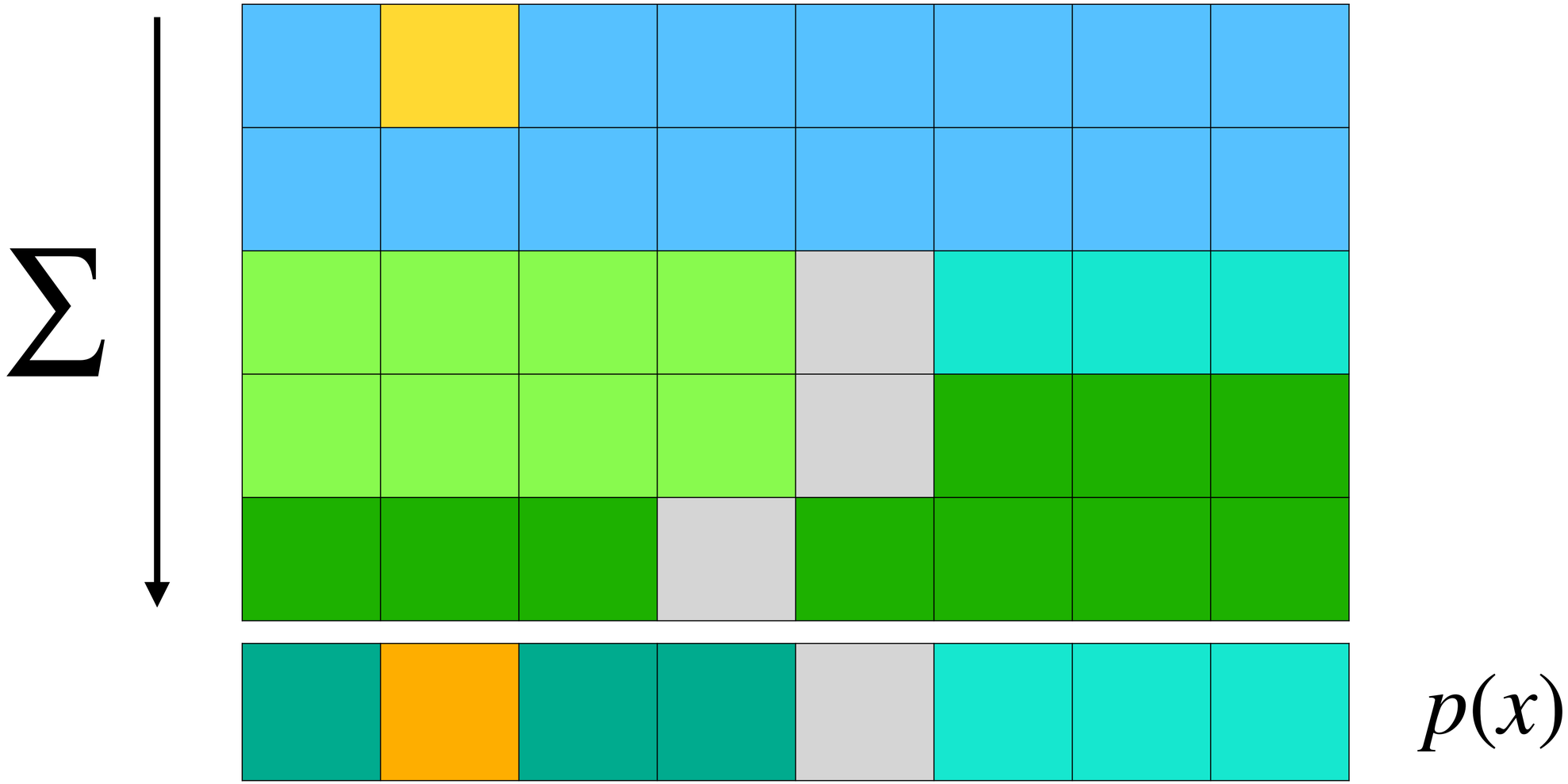
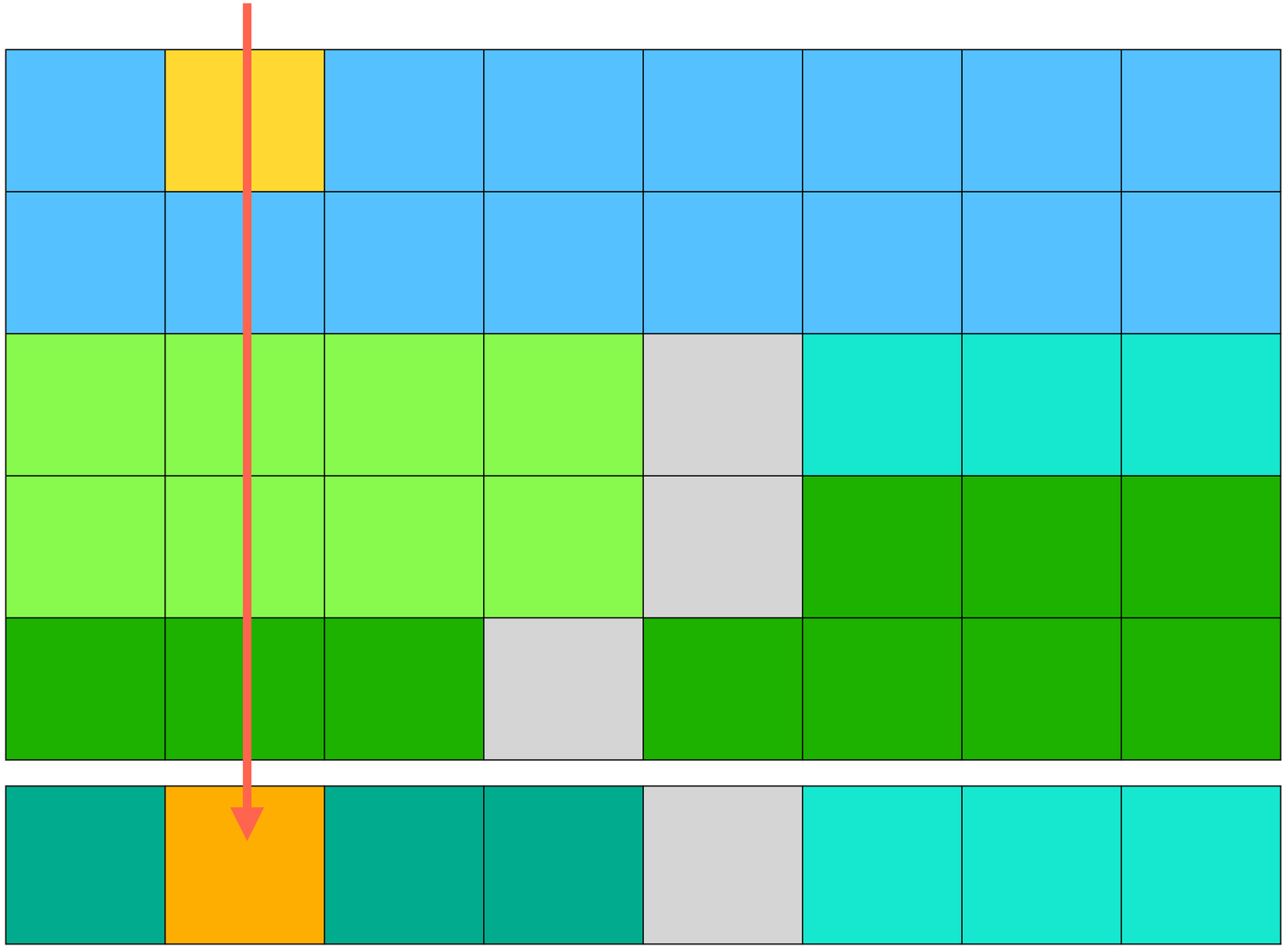
Complex Light Sources



Path-Tracing

Complex Light Sources

$$X \sim p(x)$$



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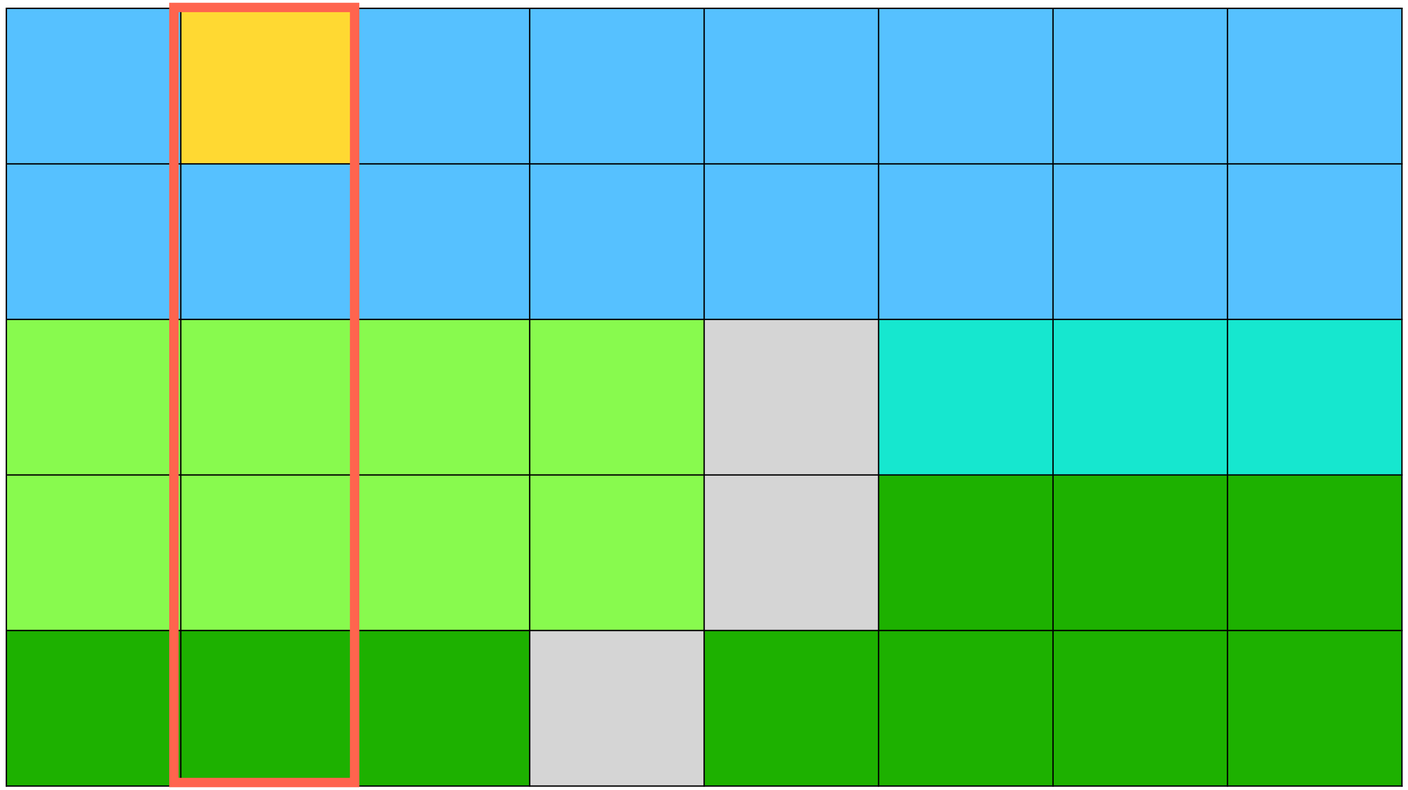
Complex Light Sources



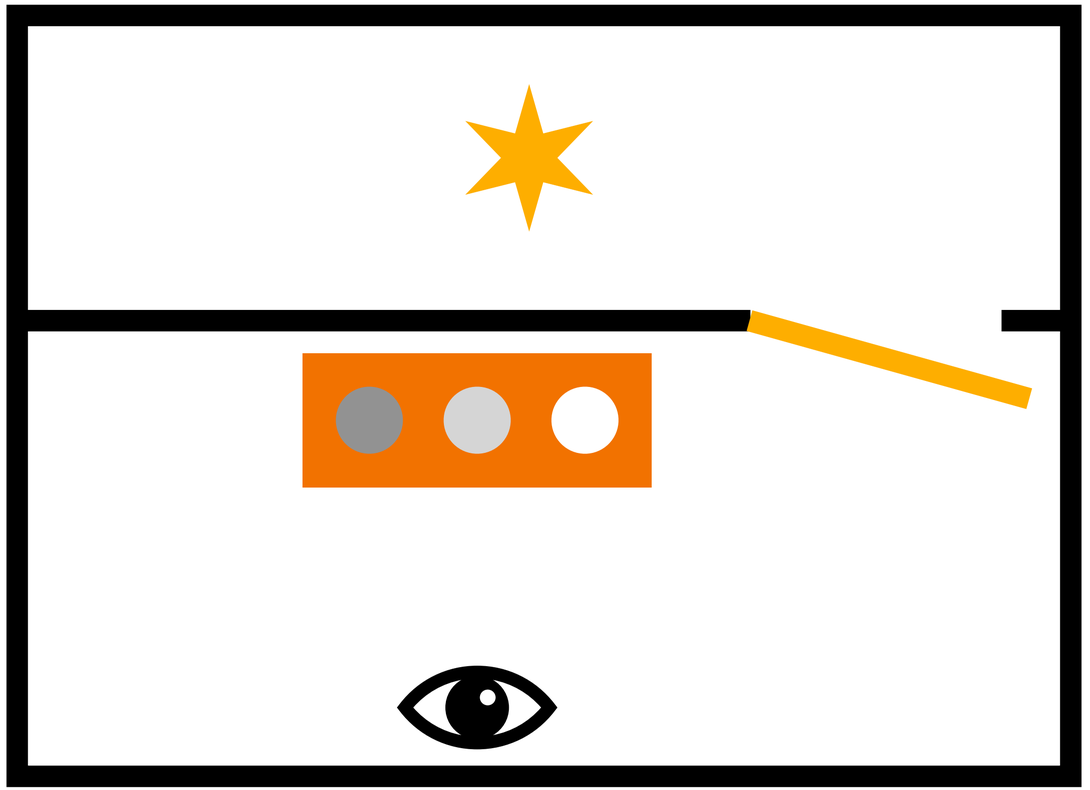
$$p(y \mid x)$$

Path-Tracing

Complex Light Sources: Example

Path-Tracing

Metropolis Sampling



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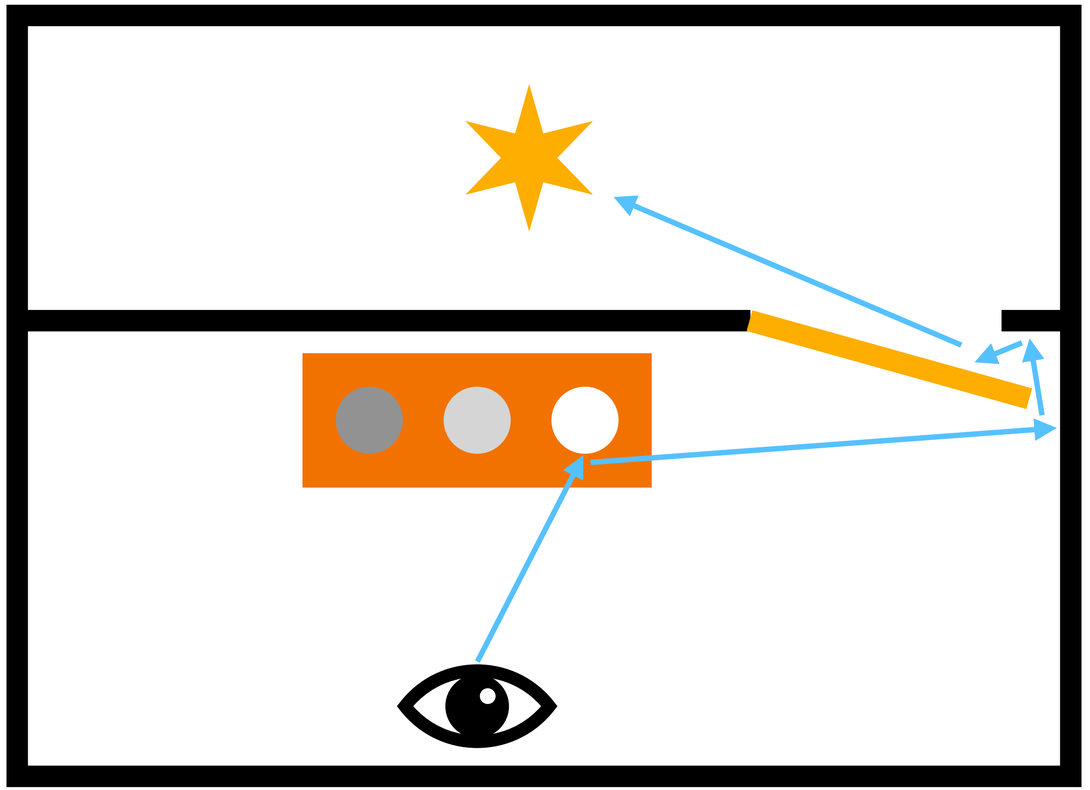
Metropolis Sampling



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Metropolis Sampling

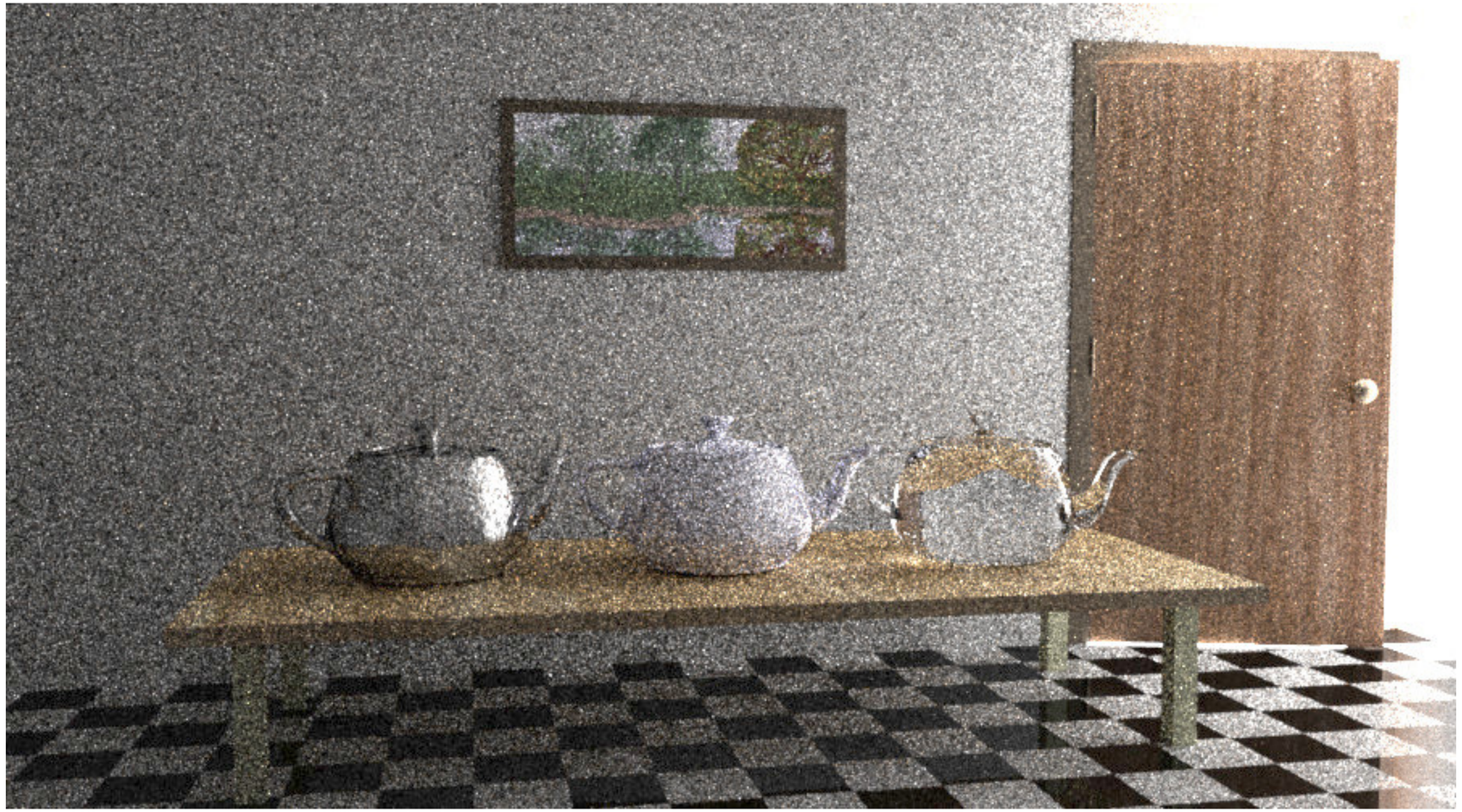
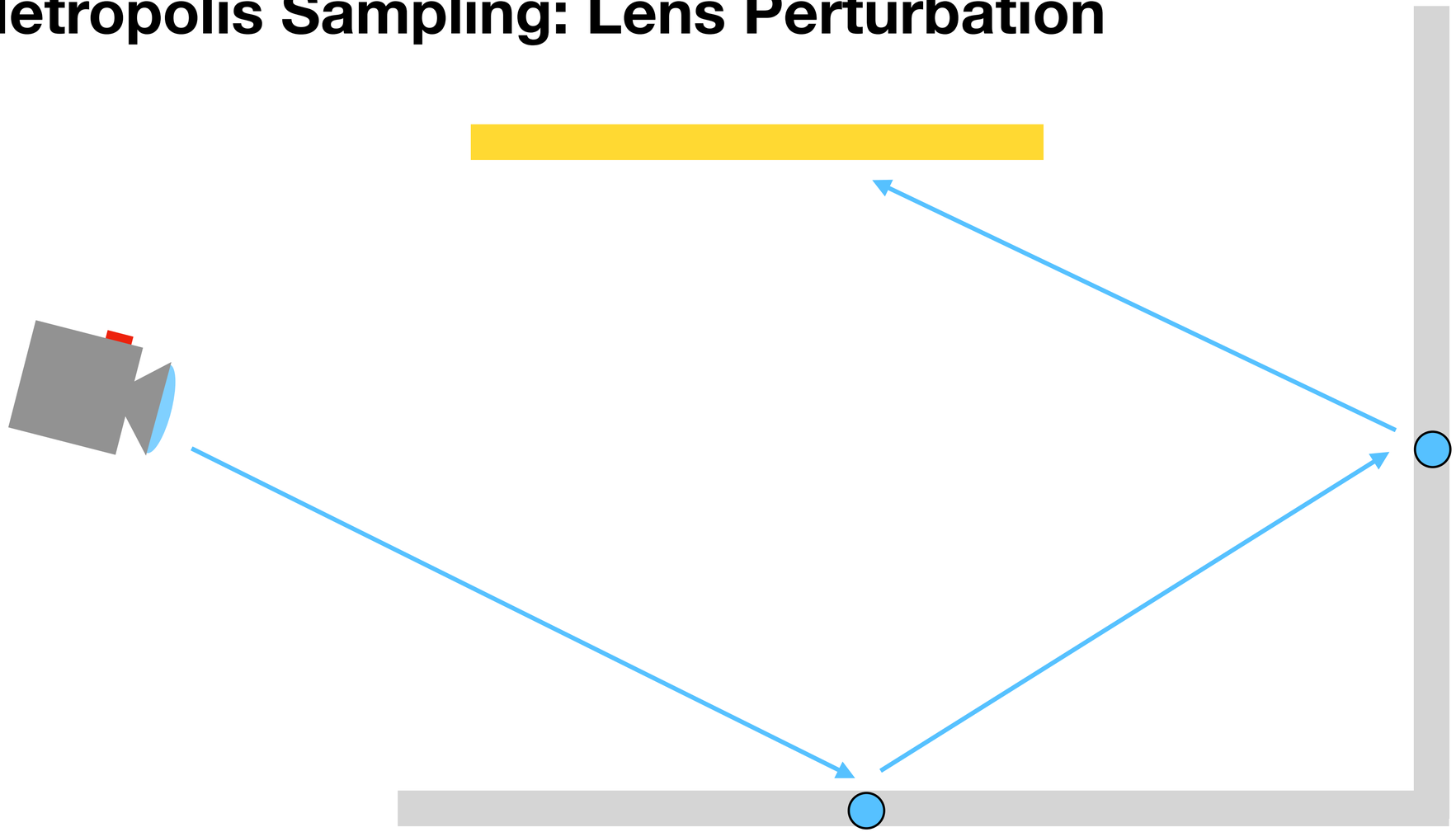


Image by Eric Veach

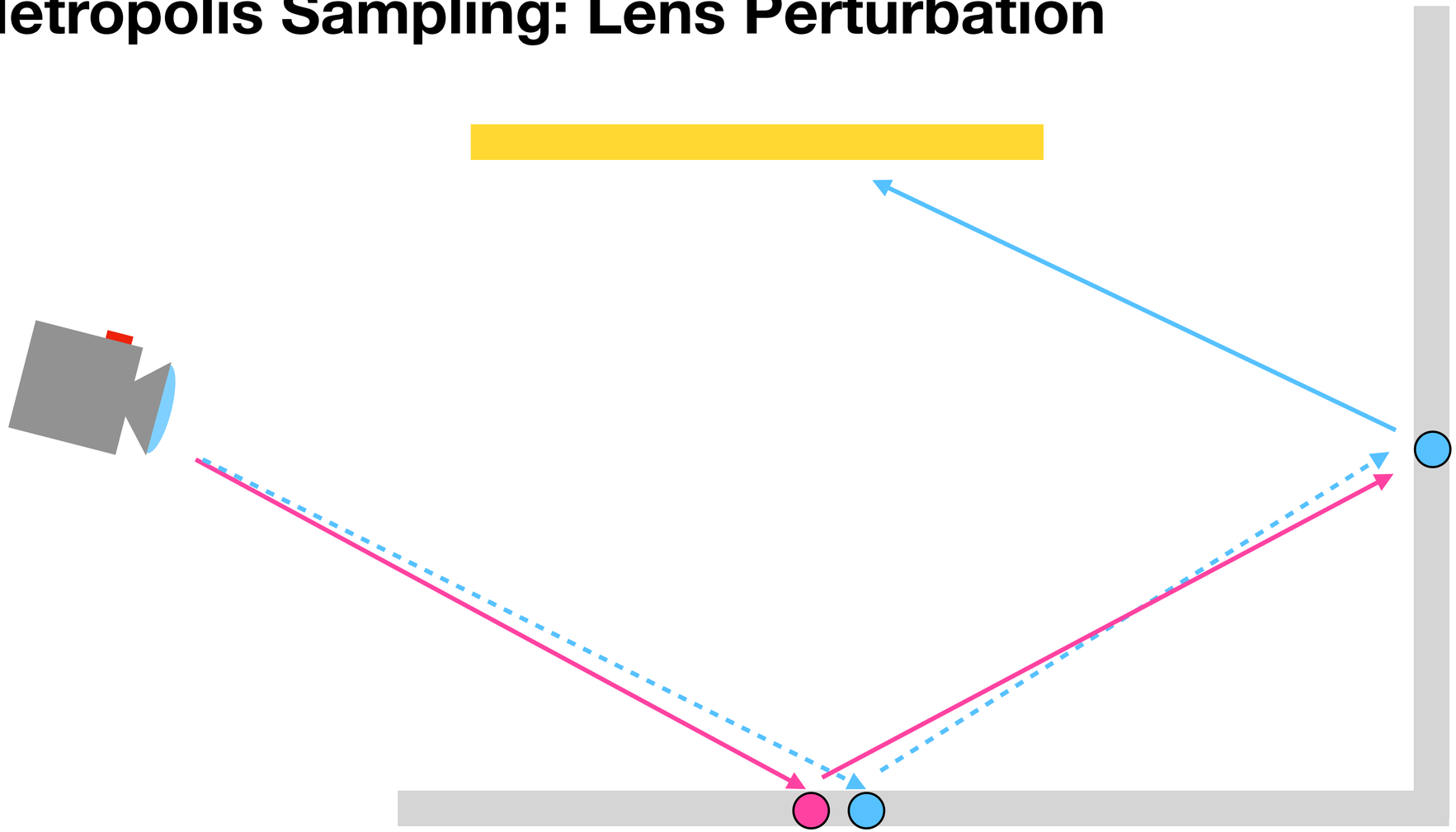
Path-Tracing

Metropolis Sampling: Lens Perturbation



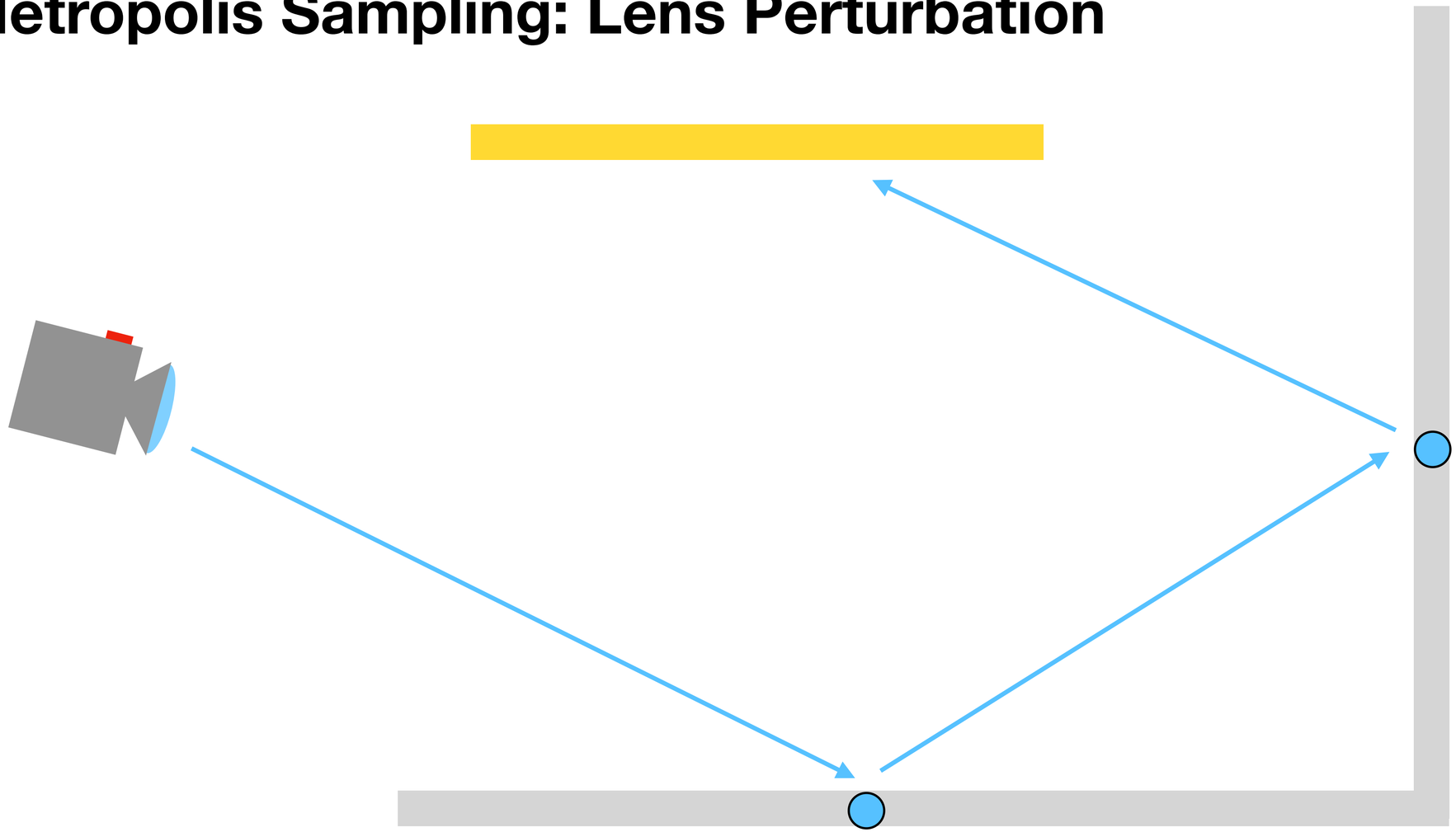
Path-Tracing

Metropolis Sampling: Lens Perturbation



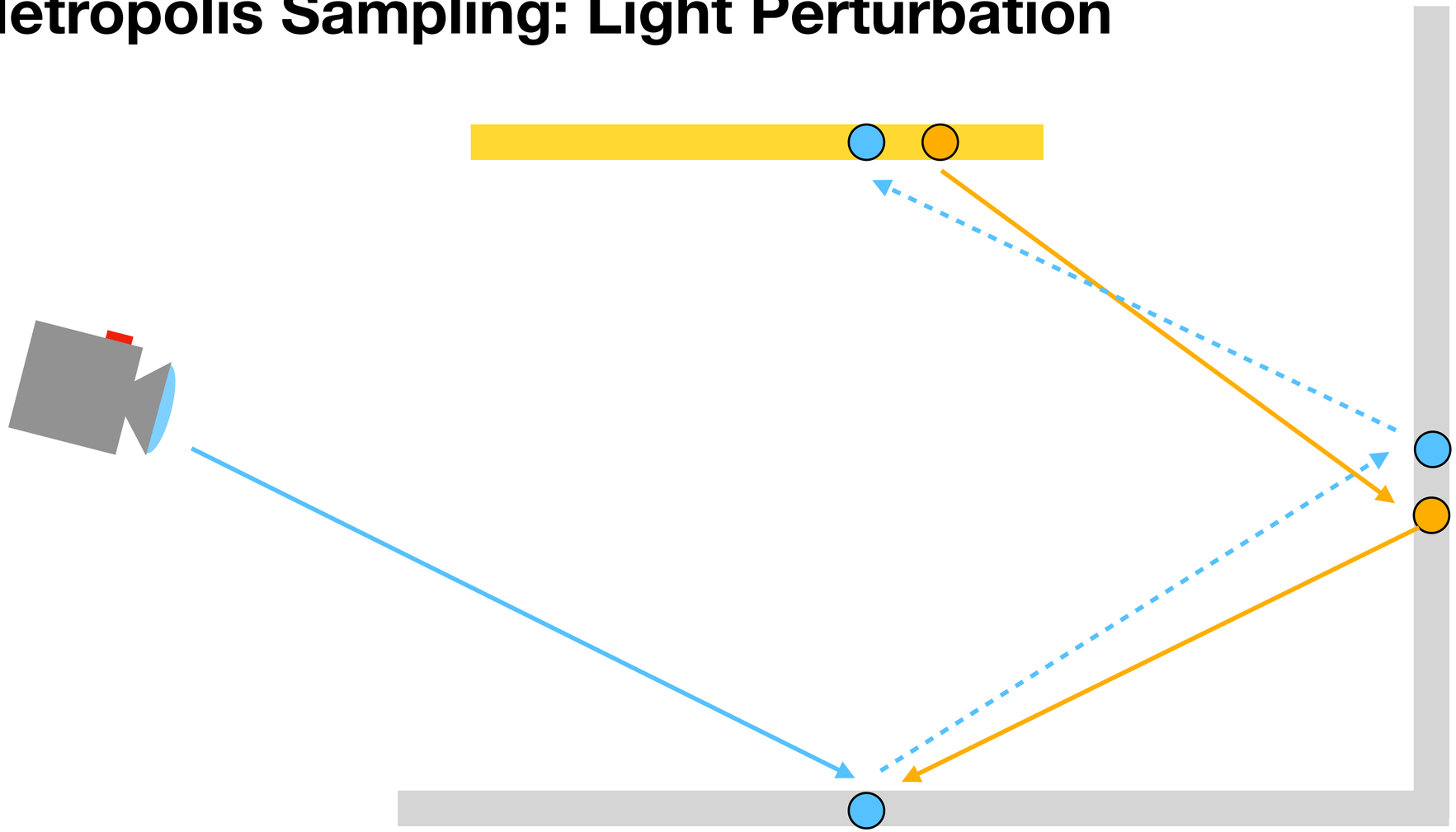
Path-Tracing

Metropolis Sampling: Lens Perturbation



Path-Tracing

Metropolis Sampling: Light Perturbation



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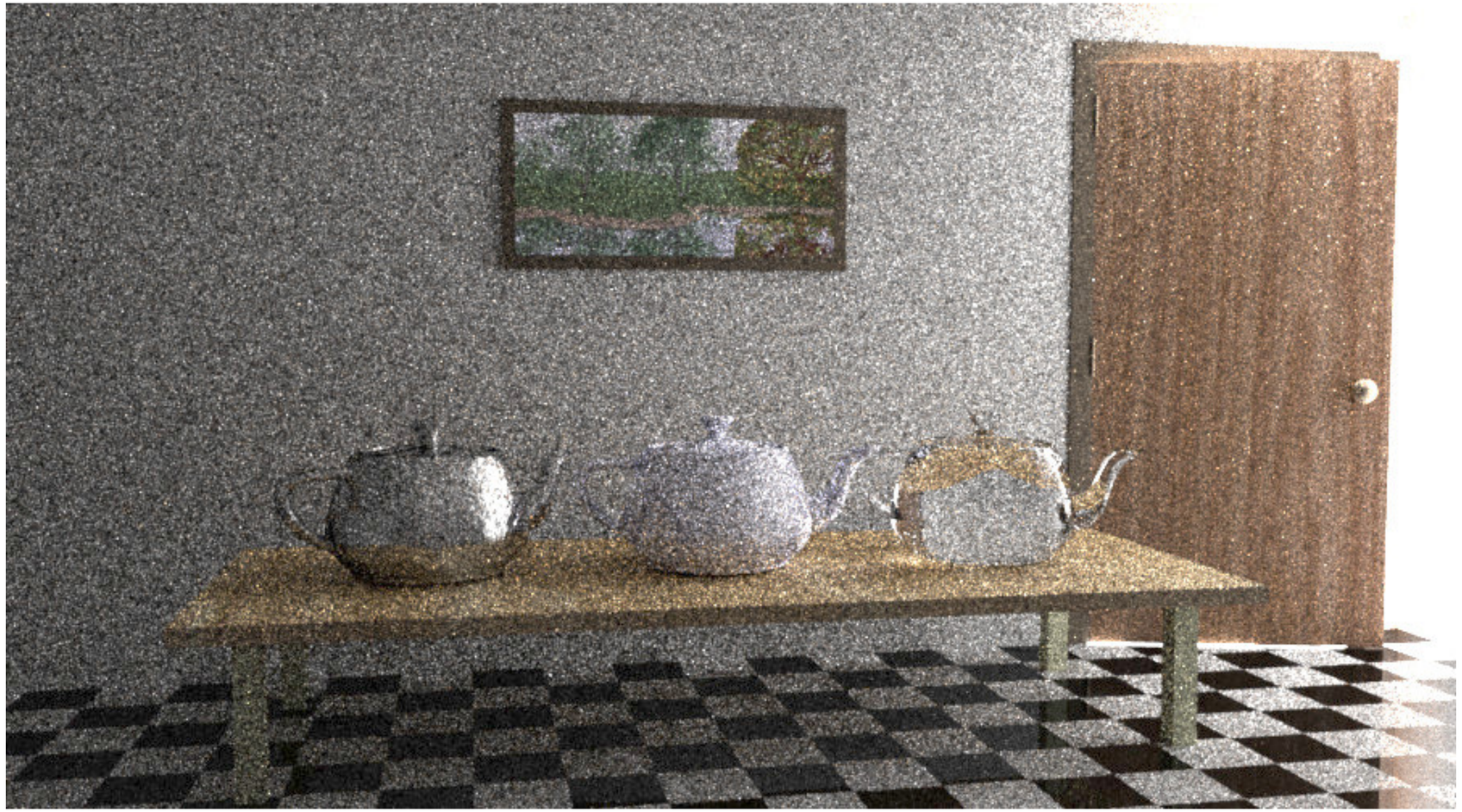
Metropolis Sampling



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Path-Tracing

Metropolis Sampling

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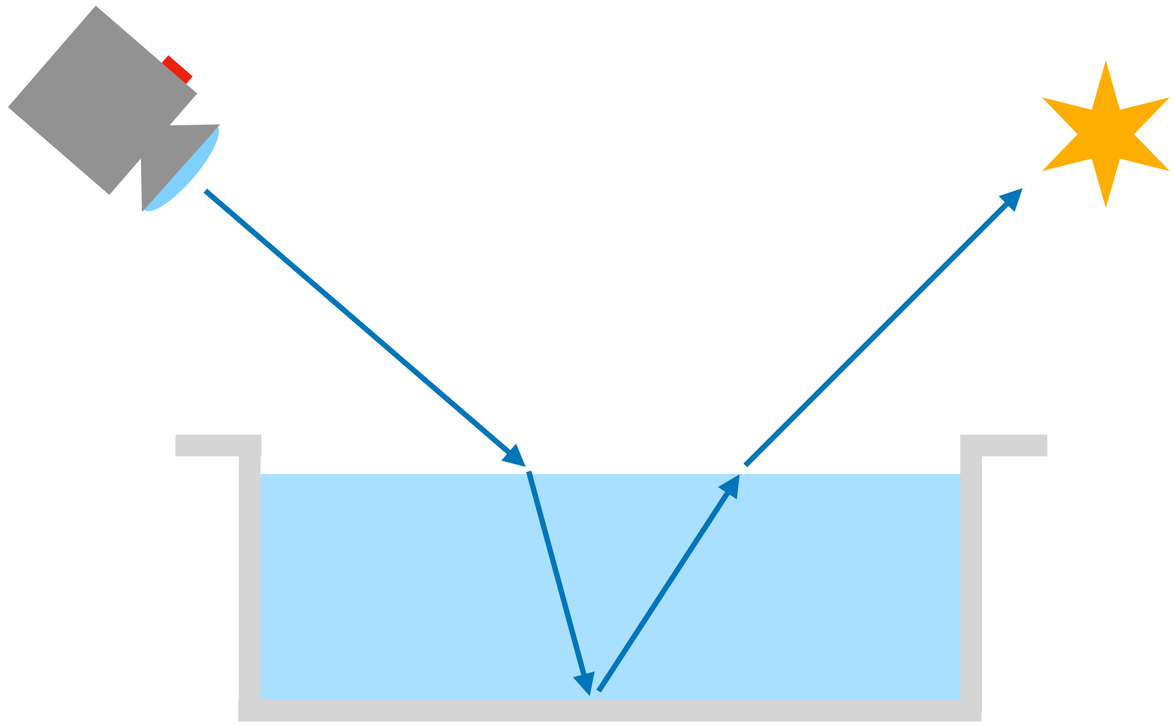
Path-Tracing

Metropolis Sampling: Multi-Chains



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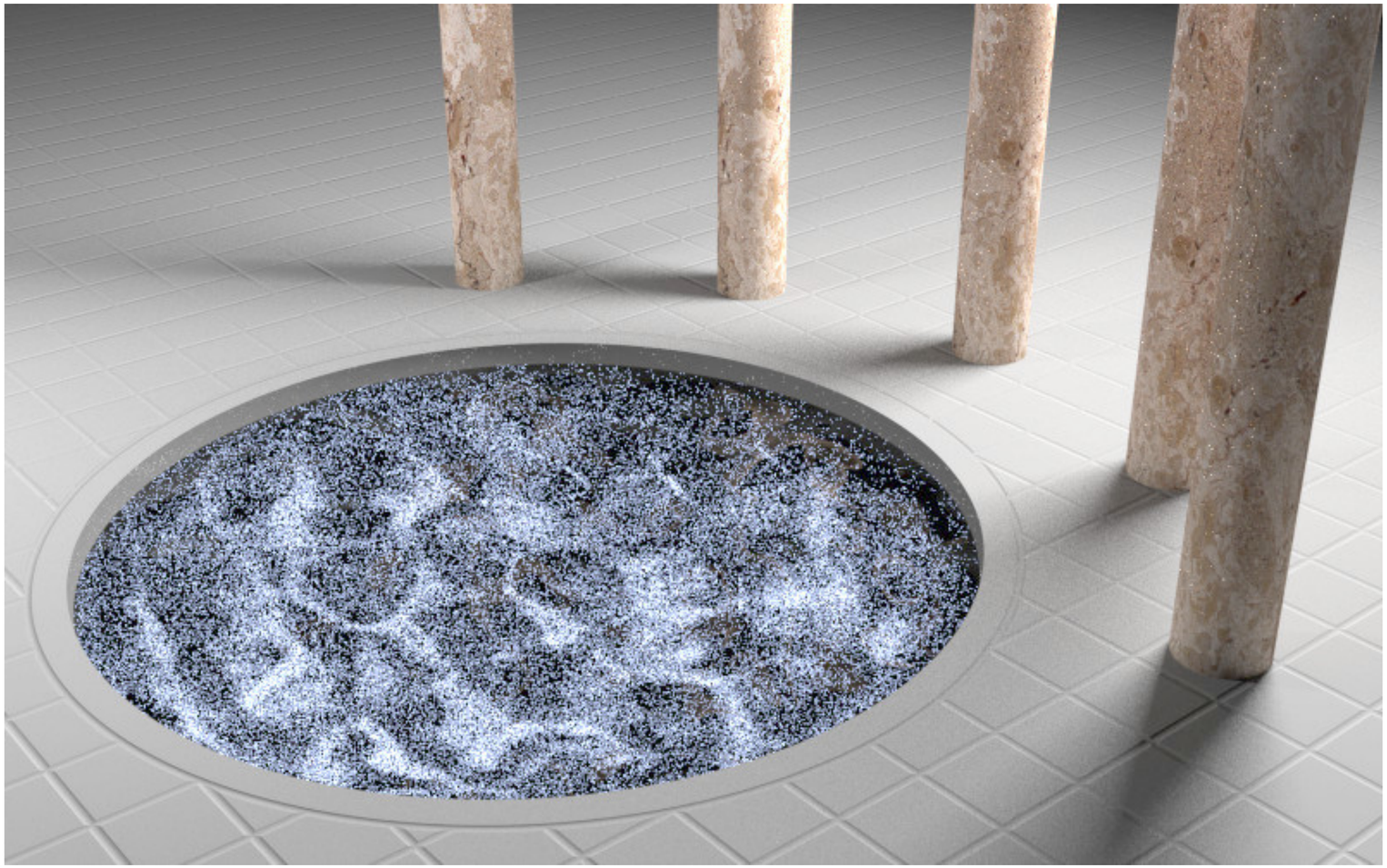
Metropolis Sampling: Multi-Chains



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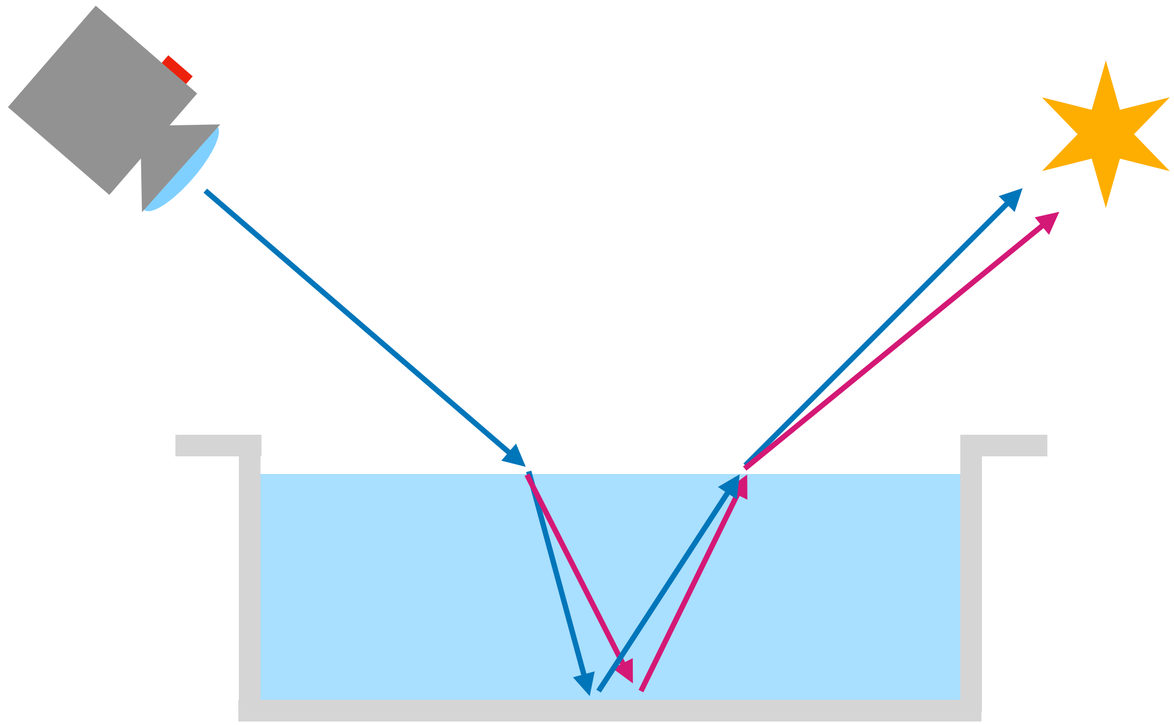
Path-Tracing

Metropolis Sampling: Multi-Chains



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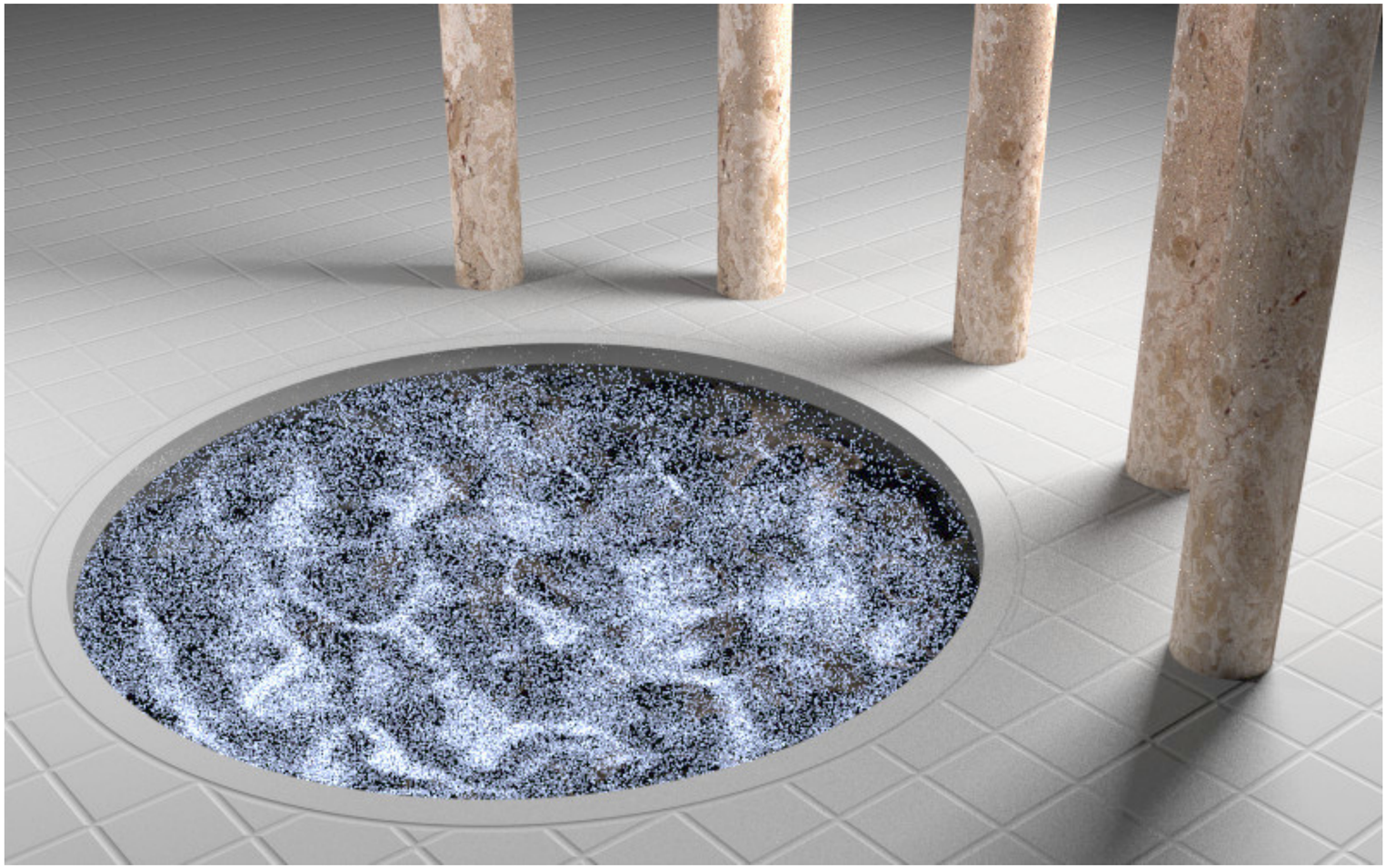
Metropolis Sampling: Multi-Chains



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Metropolis Sampling: Multi-Chains

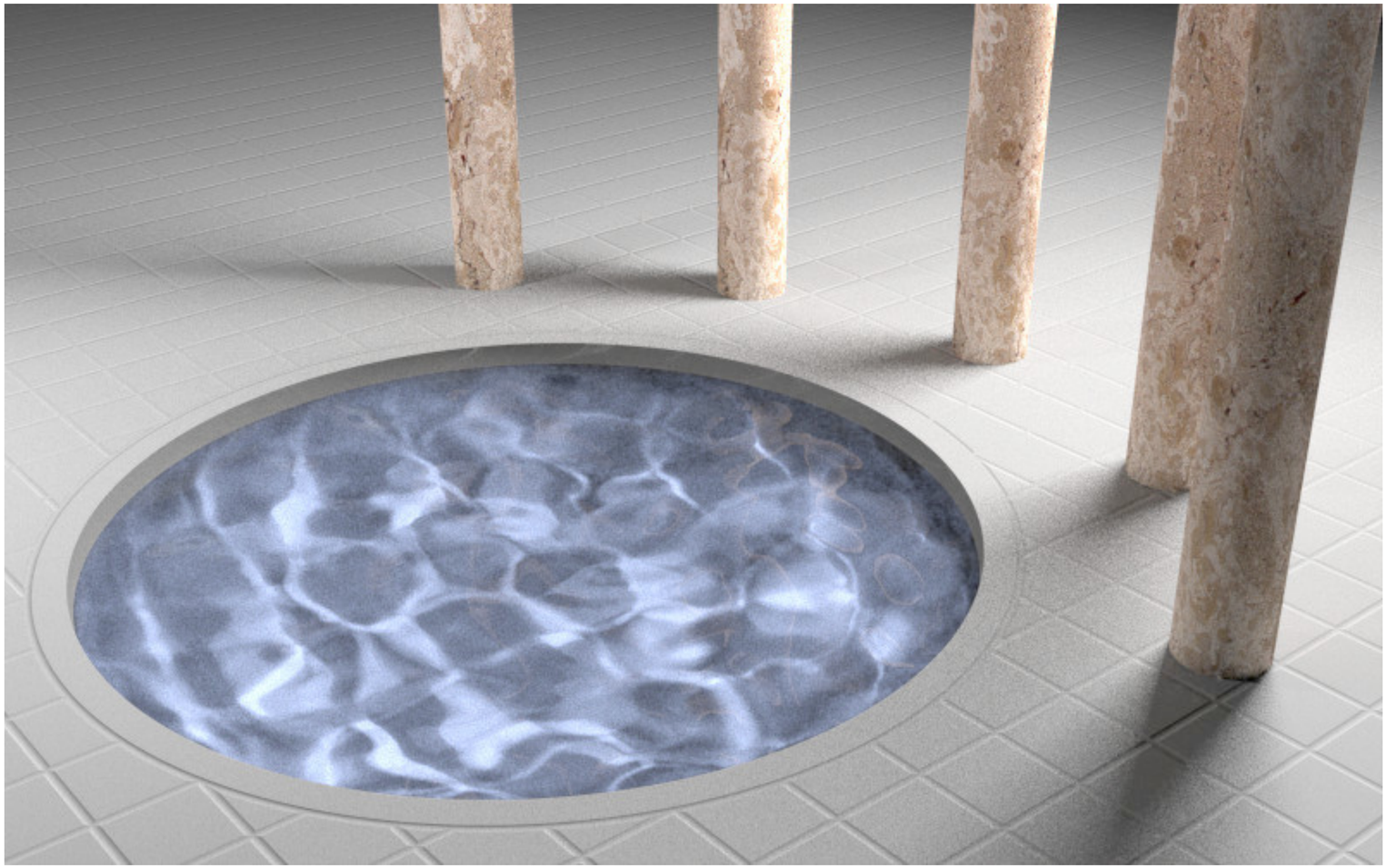

Image by Eric Veach

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Thank you for your attention!