# Monte-Carlo Methods and Sampling for Computing Monte-Carlo Integration

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# **Monte-Carlo Integration Quadrature Rules**

- Quadrature rules are efficient for 1D smooth functions:
  - Midpoint rule;
  - Trapezoidal rule;
  - Simpson rule;
  - Gauss rule;
  - etc.  $\bullet$





$$(p-a)\frac{l+1}{n}$$



$$(b-a)\frac{i+1}{n}$$



### **Monte-Carlo Integration Quadrature Rules: Trapezoidal Rule**



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## **Monte-Carlo Integration Quadrature rules**

• In general, all quadrature rules can be distilled into the following equation:

where  $w_i$  are the weights of a polynomial.

- The error is  $O(n^{-r})$  where r depends on:
  - The quadrature rule.
  - How smooth f is.

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

# **Monte-Carlo Integration** The Curse of Dimensionality

- However, when we have multidimensional integrals, quadrature methods generate samples on a multidimensional grid:
  - Reduced accuracy!
- Let's see a quadrature rule for a d-dimensional integral:

$$\int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} f(x_1, \dots, x_d) dx_1 \dots dx_d \approx \sum_{i_1=1}^{n_1} \dots \sum_{i_1=d}^{n_d} \left( \prod_{j=1}^d w_{ji_j} \right) f(x_{i_1}, \dots, x_{i_d}).$$

• Note that we apply a 1-dimensional rule for each dimension. Therefore the error becomes:

• Note that increasing r would not help much.

$$O\left(n^{-\frac{r}{d}}\right)$$

### **Monte-Carlo Integration** Basics

- Our goal is to compute this integral:  $I = \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x}, \text{ where } f : \mathbb{R}^d \to \mathbb{R}$
- We defined a new function:

$$g(\mathbf{x}) = \frac{f(\mathbf{x})}{p(\mathbf{x})}$$

(-), and p is a PDF.

### **Monte-Carlo Integration Basics**

• The expected value of  $g(\mathbf{x})$  is:

$$\mathbb{E}[g(\mathbf{x})] = \mu = \int_{\mathbb{R}^d} g(\mathbf{x}) d\mathbf{x}$$

• We approximate it as:

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n g(\mathbf{x}_i) = \frac{1}{n}$$

 $\int_{\mathbb{D}^d} g(\mathbf{x}) p(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{D}^d} f(\mathbf{x}) d\mathbf{x}.$ 

 $\frac{1}{n} \sum_{i=1}^{n} \frac{f(\mathbf{x}_i)}{p(\mathbf{x}_i)} \qquad \mathbf{x}_i \sim^{\text{i.i.d.}} p.$ 

- Let's compute the integral of  $f(x) = -x^2 + 2$  in the interval [0,2]:
- As first step, we need to define a PDF to use.
- that interval or [a, b] in general, p is going to be uniform:

p(x) =

• Now let's draw random samples according to p(x):

$$x_i = (a - b)u_i \cdot$$

• Since we need to compute the integral in [0,2] and we can draw uniform samples in

$$\frac{1}{b-a} = \frac{1}{2}.$$

 $+b \quad u_i \in \mathbf{U}(0,1).$ 

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 $\int_{0}^{2} (-x^{2}+2)dx \approx \frac{1}{N} \sum_{i=1}^{n} \frac{f(x_{i})}{p(x_{i})} = p(x) = \frac{1}{2}$  $\frac{1}{N}\sum_{i=1}^{n}\frac{(-x^2+2)}{\frac{1}{2}} = \frac{1}{N}\sum_{i=1}^{n}\frac{(-x^2+2)}{\frac{1}{2}} = \frac{2}{N}\sum_{i=1}^{n}(-x_i^2+2)$ 

- **1.3**xxx, with only two stable decimals.
  - Note that the result of this integral is  $-\frac{1}{2}$ .
  - This is not a great performance.
  - How many samples do we need?

• In this simple case, if we draw 100,000 samples we will get as integral value

• We can compute  $\sigma_0^2$  for g(x) = f(x)p(x):

$$Var(g(x)) = \mathbb{E}(g(x)^2) - \mathbb{E}(g(x))^2 = \int_0^2 g(x)^2 p(x) dx - \left(\int_0^2 g(x) p(x) dx\right)^2$$
$$= \int_0^2 \left(2(-x^2+2)\right)^2 \frac{1}{2} dx - \left(\int_0^2 \left(2(-x^2+2)\right) \frac{1}{2}\right)^2 = \frac{256}{45} = 5.689$$

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• So, with an error  $\epsilon = 0.01$ , and a confidence level  $\alpha$  at 99%, we need:

$$n \ge \frac{\sigma_0^2}{\epsilon^2} \frac{1}{1 - \alpha} = \frac{5.6}{0.6}$$

 $\frac{689}{.01^2} \frac{1}{1 - 0.99} = 5,689,000.$ 

## **Monte-Carlo Integration** Conclusions

• Quadrature Rule Error:

• Monte-Carlo Error:





# Bibliography

- Art Owen. "Chapter 1: Introduction" from the book "Monte Carlo theory, methods and examples". 2013.
- Art Owen. "Chapter 2: Simple Monte Carlo" from the book "Monte Carlo theory, methods and examples". 2013.
- Art Owen. "Chapter 7: Other Integration Methods" from the book "Monte Carlo theory, methods and examples". 2013.
- Peter Shirley, Changyaw Wang, Kurt Zimmerman. "Monte Carlo Techniques for Direct Lighting Calculations". ACM Transactions on Graphics. Volume 15. Issue 1. Jan. 1996.

Thank you for your attention!